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Analysis of nonstationary inventory systems

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I. INTRODUCTION AND REVIEW OF LITERATURE

Consider management activities associated with efficient raw material handling, maintenance (or repair) policy establishment, work-in-process material handling, production cost reduction, economic plant design and layout, distribution and warehousing of finished products, resource allocation, sales cost reduction, budget control, etc. All these managerial activities may be performed based on the analysis of demand or supply rates, which may change with time, and which should explicitly be considered in inventory management. For example, the price of some raw material used by manufacturers or the demand rates of customers for some products may exhibit considerable fluctuation in a seasonal pattern, and realistic inventory models must account for this uncertainty in demand.

"When to order" and "How much to order" are two fundamental questions involved in every inventory system. Inventory systems are largely divided into two groups, according to whether any managerial control over demand or resupply is possible. One group of inventory systems operates under essentially controllable demand. Most businesses and military inventory systems come under this category. On the other hand, the resupply of water into dams, for example, is not controllable. This study is concerned with the first group of inventory systems, i.e., those which exhibit some freedom in the determination of when, and in what quantity, the inventory should be replenished. In particular, this thesis is concerned with minimizing the cost of maintaining inventories, while at the same time keeping a sufficient stock on hand to meet contingencies arising from random demand and lead time delay.

Inventory systems are operated largely based on some operating policies concerning review systems and ordering rules. The so-called transactions-reporting systems and periodic-review systems are commonly used for inventory system review. When transactions reporting is used, all transactions of interest (for example, demand, placement of order, receipt of shipment, etc.) are recorded as they occur, and the information is immediately made known to the decision maker. For example, it may be possible to make decisions concerning the operation of the system, such as the decision whether or not to place an order, each time a demand occurs. Though it may be costly and difficult to use a reporting system of this type, there are benefits to be gained if it is not too costly, because, among other things, it may be possible to cut down on the average investment in inventory by doing so. On the other hand, in the periodic-review systems an order can be placed only at a review time with corresponding savings in the operation of the inventory system, but with likely additional penalties in inventory holding and backorder costs.

Some examples of operating doctrines are the so-called <Q, r> <R, r>, <R, T>, <nQ, r, T> and < R, r, T> identified in the book, Analysis of Inventory Systems, written by Hadley and Whitin (1963), where Q is an order quantity, R and r are certain control limits on inventory level, and T is a review period. Among those five doctrines, the <Q, r> and <R, r> doctrines are associated with transactions reporting, and the other three are associated with periodic review.

In particular, the $<R, r>$ model is used for transactions reporting with two inventory control levels r and R $(R > r)$ such that, if the inventory level falls to x, $x \leq r$ on some demand, we order up to the level R , i.e., a quantity $R - x$ is ordered. Such doctrine is referred to as an "Rr" doctrine.

The $<Q, r>$ model is a special case of $<R, r>$ model with $R = r + Q$. With the model, an order is placed when the inventory level reaches the reorder point r . Therefore, it is necessary to examine the system after every demand. It is sometimes called a continuous review system.

The $<R, T>$ model is called an "order up to R" doctrine with a review time period T . An order should be placed at each review time if there have been any demands at all in the past period. A sufficient quantity is ordered to bring the inventory position or the amount on hand plus on order up to a level R . With this system, the quantity ordered can vary from one review period to the next one.

The $<R, r, T>$ model is referred to as an "Rr" rule, which makes a procurement at a review time only if the inventory position or the amount on hand plus on order is less than or equal to r , where the inventory position is defined to be the amount on hand plus on order minus backorders. The "order up to R" rule is a special case of an "Rr" rule in which $r = R - 1$ when the inventory levels are treated as discrete variables, and $r = R$ when they are treated as continuous variables.

The <nQ, r, T> model is a "nQ" doctrine. The quantity ordered is chosen to be an integral multiple of some fundamental quantity Q; i.e., nQ for integer n. A procurement is made at a review period only if the inventory position or the amount on hand plus on order at the review time is less than or equal to r. It may not get the inventory position reached up to a level R. After the order is placed the appropriate inventory level is less than or equal to $R = r + Q$. It will be observed that when the inventory levels are treated as discrete variables, then an "order up to R" rule is a special case of the "nQ" rule for which $Q = 1$ and $R = r + 1$. When the inventory levels are treated as continuous variables, it is still true that the "order up to R" rule is a special case of the "nQ" doctrine in the limit as $Q \to 0$.

One approach to inventory system analysis is to optimize some or all of the parameters r, Q, R, and T, given a particular review system and ordering rule, say of one of the types above. The objective function for such optimizations typically is a suitable average inventory cost, depending on parameters such as Q, R, r, T, as well as on a set of relevant unit costs. The details of the computation of this average inventory cost, whether "ensemble" or "time", will be determined by what is assumed about the stochastic process modeling the generation of demands. Such a stochastic process is a description of a random phenomenon changing with time. In fact, it is defined to be a family of random variables. Therefore, the family of random demands, say

$\{N_t; t \in T\}$ with the index set T, is a stochastic process, where $\{N_t\}$ represents the cumulative demand by time $t \geq 0$. The assumptions concerning N_t once made, one may infer the relevant properties of the so-called "Inventory Position Process $\{IP_t; t \geq 0\}$," and thence the relevant properties of the so-called "Net Inventory Process $\{NIS_t; t \geq 0\}$," from which, finally, the cost process is derived whose average we seek, where the net inventory is defined to be the amount on hand minus backorders.

In consideration of a continuous-review inventory system with backorders, Galliher, Morse and Simond (1958), and Hadley and Whitin (1963) have shown that under the $<Q, r>$ model the limiting distribution of inventory position $\{IP_t; t \geq 0\}$ is uniform on the set $\{r+1, r+2, \ldots, r+Q\}$, when the interarrival times $\{X_i; i = 1, 2, \ldots\}$ between successive demands are independently and identically distributed (iid) random variables possessing negative exponential distribution and units are demanded one at a time.

Under the slightly modified replenishment policy $<nQ, r>$, Simon (1968) has also achieved the same result for the demand process in which the demand quantity is random, lead times are arbitrarily distributed, and backorders are allowed. However, the $<nQ, r>$ model has been studied under the assumption of stationary demand process, and it functions in the same manner of the $<nQ, r, T>$ periodic-review model operation with the varied review period T.

Sivazlian (1974) has generalized the work done by Galliher, Morse and Simond (1958), and Hadley and Whitin (1963). With the restriction that units be demanded one at a time, he has shown that the limiting distribution of inventory position is uniform over the set $\{r+1, r+2, \ldots, r+Q\}$ and hence is independent of the distribution of the iid interarrival times $\{X_i; i = 1, 2, \ldots\}$.

Richards (1975) seems to suggest that the result of Sivazlian is a special case of the result given by Simon. In addition, he showed that in the case of random demand quantity the limiting distribution is not uniform under the $<Q, r>$ policy.

It is known that the application of the Markov Chain Theory to inventory system analyses has the advantage of yielding directly the state probabilities of inventory positions so that the average annual cost can be easily determined. Some discrete-parameter stochastic processes $\{X_t; t = 0, 1, 2, \ldots\}$ have the outcome functions $\{X_t(\omega)\}$ with $\omega \in \Omega$ (sample space) which range over the elements of a countable state space $S = \{1, 2, \ldots\}$. Therefore, a finite discrete-parameter stochastic process has the outcome functions $\{X_t(\omega); \omega \in \Omega\}$ which range over the elements of a finite state space $S = \{1, 2, \ldots, N\}$. A discrete-time Markov chain is a stochastic process $\{X_t; t = 0, 1, 2, \ldots\}$ possessing the state space $S = \{1, 2, \ldots\}$ or $S = \{1, 2, \ldots, N\}$ and satisfying the Markov property that the future state of the system is determined according to transition probabilities depending only on the current state of the system. In other words, a sequence of states chosen by such stochastic process forms a discrete-time Markov chain. If the transition probabilities change with time, then the

Markov chain is called nonstationary. Otherwise, it is called stationary.

In the case of periodic-review inventory systems, Hadley and Whitin (1963) have applied stationary Markov Chain Theory to find the limiting distributions of inventory positions $\{IP_{T_k} ; T_k \geq 0$ for $k = 0, 1, 2, \ldots\}$ (where T_k is the k^{th} review time) with a constant review interval T such that $T \equiv T_{k+1} - T_k$ for all k, and finite state spaces $S = \{r+1, r+2, \ldots, r+Q\}$ and $S = \{r+1, r+2, \ldots, R\}$ for the $\langle nQ, r, T \rangle$ model and the $\langle R, r, T \rangle$ model, respectively.

Veinott (1965) studied on the nonstationary periodic-review inventory problems with arbitrary demand process in a very general manner. He did not investigate the specific structure of the relation between $\{IP_{T_k}\}$ and $\{D_{(T_k, T_k+\xi]} ; \xi \geq 0\}$ $(k = 0, 1, 2, \ldots)$ and the sufficient conditions for the existence of the limit distribution of $\{IP_{T_k}\}$. Rather he worked on determining optimal policies under the assumption of independent random inter-period demands.

None of the above authors considered the possibility of the application of the nonstationary Markov Chain Theory to the periodic-review inventory models with nonstationary (or nonhomogeneous) demand process.

A. Research Objective

The primary objective of this study is to analyze nonstandard inventory models, with general independently and identically distributed (iid) inter-demand times for transactions reporting, and nonstationary Markov demand for periodic review.

This subject will be developed in the context of the case in which demands occurring when the system is out of stock are backordered, units are demanded one at a time, and procurement lead time is constant. Moreover, the inventory system under study will consist of just one stocking point with a single source for resupply.

Under the above assumptions, the cumulative demand by time t, $\{N_t;\ t \geq 0\}$, is a discrete-valued continuous-parameter stochastic process with sample paths increasing in unit steps. $\{N_t\}$ will be analyzed to describe probabilistically the inventory position $\{IP_t;\ t \geq 0\}$, under the $\langle Q, r\rangle$ model for transactions reporting, and under the $\langle nQ, r, T\rangle$ and $\langle R, r, T\rangle$ models for periodic review.

During the process of analyzing $\{N_t;\ t \geq 0\}$ under the $\langle Q, r\rangle$ model for transactions reporting in Chapter II, it will be shown that the inventory position $\{IP_t;\ t \geq 0\}$ totally depends upon the demand process $\{N_t;\ t \geq 0\}$. For example, if an inventory system is started with $IP_0 = r + i$ $(i = 1, 2, \ldots, Q)$ at time $t = 0$, then $IP_{t-\tau} = r + j$ $(j = 1, 2, \ldots, Q)$ at time $t-\tau > 0$ can be reached after the $(i-j)^+$ or $\{i + (m-1) \cdot Q + (Q-j);\ m = 1, 2, \ldots\}$ demand materialization by time $t-\tau$, where τ is a constant procurement lead time, m denotes the total number of order placements by time $t-\tau$ and $(i-j)^+ = \max(0, i-j)$. In other words, $P\{IP_{t-\tau} = x\}$ is a function of $P\{N_{t-\tau} = y\}$, as $\{IP_{t-\tau}\}$ is determined by $\{N_{t-\tau}\}$. In spite of the relation, we shall prove that given $IP_0 = r + i$ $(i = 1, 2, \ldots, Q)$ at time $t = 0$, $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ are mutually independent of each other (where $D_{(t-\tau,t]}$ is a lead time demand and so

$D_{(t-\tau,t]} \equiv N_t - N_{t-\tau}$), even in the case of nonstandard (non-Poisson) inventory models with general iid inter-demand times. If the inter-arrival times are exponentially distributed, which is known as a memoryless process, then the above independency follows. However, it may not be so obvious for the case where the inter-arrival times are generated from other types of distributions.

Nobody has proved the above independency yet. With its proof, the analysis of net inventory process $\{NIS_t;\ t \geq 0\}$ will become straightforward, from which the cost process can be immediately derived whose average we seek. Therefore, the joint distribution of $\{IP_{t-\tau}\}$ and $\{D_{t-\tau,t]}\}$ will be determined first to find the distribution of $\{NIS_t\}$ needed for the expected annual cost analysis, where by definition $NIS_t \equiv IP_{t-\tau} - D_{(t-\tau,t]}$ with $t \geq \tau \geq 0$.

The asymptotic limit distributions of $\{IP_{t-\tau}\}$, $\{D_{(t-\tau,t]}\}$ and $\{NIS_t\}$ will also be evaluated in the chapter. By use of the direct Laplace-Stieltjes Transform approach and Key Renewal Theorem (see Smith (1958) and Takacs (1958)), these limiting distributions will be determined.

It is known that the limiting behavior of a distribution function $F(t)$ can be found from the equality

$$\lim_{S \to 0^+} S\, L\{F(t)\} = \lim_{t \to \infty} F(t) ,$$

where $L\{F(t)\}$ is denoting the Laplace transform of $F(t)$ such that

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt \quad \text{for} \quad s > 0 .$$

For completeness, our proof of the equality will be presented. (See also Holl et al. (1959), and Doetsch (1961).) Then, we shall show that together with the equality, the so-called Convolution Laplace Transform Theorem, the proof of which appears in Holl et al. (1959), Doetsch (1961) and Widder (1971), can be used to get those limiting distributions. Under the assumption of the instantaneous procurement delivery, Sivazlian (1974) has considered this approach to determine the uniform limit distribution of $\{IP_t\}$ regardless of the distribution of the iid inter-demand times under the $<Q, r>$ model. However, when accounting for a positive delivery time, the Convolution Laplace Transform Theorem is not satisfactory to get the limit distribution of $\{D_{(t-\tau,t]}\}$. A corollary of the theorem is developed; for $s > 0$,

$$L\{\int_0^{t-\tau} G(t-x)\, F(x)dx\} = L\{F(t)\}\cdot\{G(t)\} - L\{F(t)\} \cdot \int_0^{\tau} e^{-sy}\, G(y)dy ,$$

$$\text{where} \quad \tau \geq 0 .$$

The limit distribution of $\{D_{(t-\tau,t]}\}$ can be more easily found by applying Key Renewal Theorem. After the long-run limit distributions of $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,]}\}$ are evaluated in Section D, then the long-run expected average annual values of on-hand inventory $E[OH]_Q$ and of backorders $E[BO]_Q$ and hence the long-run expected limit average annual cost expression will be finally derived in Section E under the assumptions of stationary cost variations.

In Chapter III, we shall first show that the process $\{IP_{T_k};\ T_k \geq 0\}$ associated with nonhomogeneous Poisson demand $\{D_{(T_k, T_{k+1}]}\}$ $(k = 0, 1, 2, \ldots)$ is a nonstationary Markov chain. Then, the nonstationary Markov Chain Theory will be applied to investigate the limiting distributions of $\{IP_{T_k}\}$ and $\{NIS_{T_k+\xi};\ \xi \geq 0\}$, where $\{T_k\}$ $(k = 0, 1, 2, \ldots)$ are the inventory system reviewing times with $T_0 \equiv 0$, and so $(T_{k+1} - T_k) \equiv \Delta T_k$ is the $(k+1)^{st}$ review period.

Dobrushin (1956) defined the ergodic coefficient α, a quantity important to the analysis of both stationary and nonstationary Markov chains. Hajnal (1956) and Mott (1957) verified conditions (implicitly in terms of the ergodic coefficient) for a nonstationary finite Markov chain to be weakly ergodic, a condition important in determining when the Markov chain is strongly ergodic and so has a long-run distribution. A Markov chain being weakly ergodic is equivalent to the Markov chain with the long-run behavior of "loss of memory without convergence," which means that the probability of being in a particular state is eventually independent of its initial state, and a strongly ergodic Markov chain has the "loss of memory with convergence" behavior. Paz (1970, 1971) extended the work of Hajnal to infinite matrices by use of a new coefficient δ which is defined to be $\delta(P) = 1 - \alpha(P)$ for a transition probability matrix P and sometimes more conveniently used. Conn (1969), Madsen and Conn (1973), and Madsen and Isaacson (1973) (Isaacson and Madsen (1976)) gave conditions in terms of left eigenvector convergence for a Markov chain to be strongly ergodic. Bowerman (1974), and Bowerman, David and Isaacson (1977) have verified sufficient conditions for the strong ergodicity of a Markov chain in which the transition matrices repeat themselves in a cyclic fashion (i.e., $P_{nd+\ell} = P_\ell$; $\ell = 1, 2, \ldots, d$; $n = 0, 1, 2, \ldots$).

In the case of the $<nQ, r, T>$ model, it will be shown that the transition probability matrices of the chain $\{IP_{T_k};\ T_k \geq 0\}_{k=0}^{\infty}$ are doubly stochastic and hence the nonstationary finite Markov Chain Theory is easily applied to determine that the long-run limit distribution of $\{IP_{T_k}\}$ is uniform under the assumption that the chain is weakly ergodic. If the transition probability matrices P_k's repeat themselves in a cyclic fashion such that $P_{nd+\ell} = P_\ell$ for $\ell = 1, 2, \ldots, d$ and $n = 0, 1, 2, \ldots$ (for example, $d = 4$ for a seasonal demand fluctuations), then the chain is weakly ergodic and hence the uniform distribution will be determined. The limit distributions of $\{IP_{T_k}\}$ and the long-run expected limit values of on-hand inventory $E[OH]_{nQ}$ and of back-orders $E[BO]_{nQ}$ will be evaluated in Section D and the corresponding cost expression will be derived in the same section.

For the $<R, r, T>$ model, the corresponding limit values of $P\{IP_{T_k} = r + j\}$ $(j = 1, 2, \ldots, R - r)$ for $T_k \geq 0$ and $k = 0, 1, 2, \ldots$, $E[OH]_R$ and $E[BO]_R$ will also be evaluated and then the cost expression will finally be derived in Section D, too. In the case of the $<R, r, T>$ model with stationary Poisson demand studied in Hadley and Whitin (1963), the simpler closed form of solutions for the long-run limit distribution of $\{IP_{kT};\ T \geq 0\}$ $(k = 0, 1, 2, \ldots)$ will be derived in Section C.

Similarly, $P\{IP_{T_{nd+\ell}} = r+j\}$ $(j = 1, 2, \ldots, R-r)$, $E[OH]_{RC}$ and $E[BO]_{RC}$ corresponding to the cyclic demand patterns under the $<R, r, T>$ model will also be analyzed in the same section to derive a cost expression.

Summary and concluding remarks are made in Chapter IV.

II. TRANSACTIONS REPORTING

A. Introduction

When the inter-arrival times of customer demands are assumed random variables, one may not know the state of an inventory system at each point in time unless each transaction (for example, demand, placement of order, receipt of shipment, etc.) is recorded and reported as it occurs. Furthermore, in the real world it may never be possible to predict customer demands with certainty; rather they had better be described in probabilistic terms.

In the transactions-reporting inventory system, all transactions of interest are recorded as they occur and the information is immediately made known to the decision maker who will determine when to order and how much to order. The so-called lot size-reorder point inventory system operating doctrine referred to as the $\langle Q, r \rangle$ model is commonly used for transactions-reporting inventory system analyses.

Under the $\langle Q, r \rangle$ model, a quantity Q is ordered each time the appropriate inventory level (for example, the on-hand inventory, the net inventory, the on-hand plus on-order inventory, or the inventory position) reaches the reorder point r, where the inventory position $\{IP_t; t \geq 0\}$ and the net inventory $\{NIS_t; t \geq 0\}$ are referred to as the amount on hand plus on order minus backorders and the amount on hand minus backorders, respectively. In fact, the inventory position is chosen as a suitable inventory level for defining the order quantity Q and the reorder point r.

Another description of the $<Q, r>$ model is given as a transactions-reporting inventory system operating doctrine under which an order is placed for the quantity Q to raise the inventory position to the level $r+Q$ as soon as a demand drops the inventory position below the level $r+1$. Thus, the inventory position successively falls from $r+Q$ to $r+1$ during each procurement cycle, and instantaneously rises again up to $r+Q$.

Under this $<Q, r>$ model, Hadley and Whitin (1963) have analyzed some transactions-reporting inventory systems with Poisson demand.

The primary objective of this chapter is to analyze the $<Q, r>$ transactions-reporting inventory system for the backorders case with general iid (independent, identically distributed) inter-demand times and constant lead time τ. The $<Q, r>$ model is known as a special case of an $<R, r>$ model with $R = r+Q$, under which an order is placed to get the inventory position up to the level R when the inventory level falls below r. This $<R, r>$ model, however, won't be covered in this study.

The subject will be developed in the context of the case in which demands occurring when the system is out of stock, are backordered, units are demanded one at a time, and procurement lead time τ is constant. Moreover, it will be assumed throughout this chapter that the inventory system consists of just one stocking point with a single source for resupply.

Under the above assumptions, the cumulative demand by time t, $\{N_t;\ t \geq 0\}$, is a discrete-valued continuous-parameter stochastic process (a renewal counting process) with sample paths increasing in unit steps, where a stochastic process is a description of a random phenomenon changing with time. $\{N_t;\ t \geq 0\}$ will be analyzed in Section B of this chapter to describe probabilistically the inventory position $\{IP_t;\ t \geq 0\}$ under the <Q, r> model. During the process of analyzing $\{N_t\}$, it will be shown that $\{IP_t;\ t \geq 0\}$ totally depends upon the demand process $\{N_t;\ t \geq 0\}$. Let $D_{(t-\tau,t]}$ denote a procurement lead time demand during the time interval $(t - \tau,\ t]$, so that $D_{(t-\tau,t]} \equiv N_t - N_{t-\tau}$.

In Sections B and C, Renewal Theory will be applied to prove that even if $\{P_t\}$ is dependent upon $\{N_t\}$, $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ for $t \geq \tau \geq 0$ are mutually independent of each other. This nature of the relation between $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ leads to the formulation of the joint distribution of $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ which can be used to determine the distribution of net inventory $\{NIS_t\}$ needed for the expected long-run average annual cost expression.

A corollary of the so-called Convolution Laplace Transform Theorem will be developed in Section B and applied to the computation of the asymptotic limit distributions of $\{IP_{t-\tau}\}$, $\{D_{(t-\tau,t]}\}$ and $\{NIS_t\}$ in Section D.

In the last Section E, we shall discuss the nature of the relevant cost factors in the inventory system. The stationary cost factors will be considered for this study. Then, the probability, say P_{os}, that the system is out of stock, the long-run expected on-hand inventory

$E[OH]_Q$, and the long-run expected backorders $E[BO]_Q$ will be determined and followed by the formulation of the long-run expected average annual cost expression to be optimized under some assumptions on cost factors. The objective cost function for such optimizations typically is a suitable average inventory cost, depending on parameters such as Q, r, T as well as on a set of relevant unit costs.

B. The Demand Process and Renewal Theory

A stochastic process is a description of a random phenomenon changing with time. From the point of view of the mathematical theory of probability a stochastic process is best defined as a family $\{X(t); t \in T\}$ of random variables, where the parameter set T is called the index set of the process. Two important cases are a discrete parameter set, e.g., $T = \{0, \pm 1, \pm 2, \ldots\}$, and a continuous parameter set, e.g., $T = \{t; -\infty < t < \infty\}$. Throughout this chapter we shall take the continuous parameter set, $T = \{t; t \geq 0\}$.

When demands arrive at time points $t_1, t_2, \ldots,$ $(0 < t_1 < t_2 < \ldots)$, the successive inter-arrival times $\{X_i; i > 1\}$ are defined as $X_1 = t_1$, $X_2 = t_2 - t_1, \ldots,$ $X_n = t_n - t_{n-1}, \ldots$ Let N_t be cumulative demand by time t , $t \geq 0$. Then $\{N_t; t \geq 0\}$ is a discrete-valued continuous-parameter stochastic process with sample paths increasing in unit steps.

Assume that demands in the inventory system occur one at a time and that the demand inter-arrival times $\{X_i; i = 1, 2, \ldots\}$ are independent identically distributed random variables with a common probability distribution F with $F(0) = 0$, since demands occur one at a time. Further, assume that the procurement lead time τ is constant and that units demanded when the system is out of stock are backordered. Then, $\{IP_t;\ t \geq 0\}$ also is a discrete-valued continuous-parameter stochastic process. Its range, however, is restricted to the integers $(r+1, r+2, \ldots, r+Q)$. The integer-valued, or counting, process $\{N_t;\ t \geq 0\}$ is a renewal counting process generated by the inter-arrival times X_i, since the successive inter-arrival times $X_1, X_2, \ldots,$ are assumed to be independent identically distributed positive random variables. Denote by S_n the renewal epoch of the n^{th} demand (the time of the n^{th} renewal), so that $\{S_n;\ n = 0, 1, 2, \ldots\}$ are the partial sums of the renewal process $\{X_i\}$, that is,

$$S_n = \sum_{i=1}^{n} X_i , \qquad (S_0 \equiv 0) . \tag{2.2.1}$$

In other words, S_n is the waiting time to the n^{th} demand, which represents the time it takes to register n demands if one is observing a series of demands occurring in time. There exists a basic relation between the counting process $\{N_t;\ t \in T\}$ and the corresponding sequence of waiting times $\{S_n\}$, namely,

$$N_t = \text{Sup}\,\{n;\ S_n \leq t\} , \tag{2.2.2}$$

so that one has

Proposition II.B.1:

For $t > 0$ and $n = 1, 2, \ldots,$

$$N_t \geq n \quad \text{if and only if} \quad S_n \leq t \ , \tag{2.2.3}$$

from which it follows that

$$N_t = n \quad \text{if and only if} \quad S_n \leq t \ \text{ and } \ S_{n+1} > t \ . \tag{2.2.4}$$

If X and Y are independent random variables, with X having distribution F and Y having distribution G, then the distribution of $X + Y$ is given by

$$P\{X + Y \leq t\} = \iint_{X+Y \leq t} dF(x)dG(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{t-y} dF(x)dF(y)$$

$$= \int_{-\infty}^{\infty} F(t-y)dG(y)$$

$$= \int_{-\infty}^{\infty} G(t-x)dF(x) \ . \tag{2.2.5}$$

Sometimes, the distribution $P\{X+Y \leq t\}$ is denoted by $F * G(t)$ which is called the convolution of $F(t)$ and $G(t)$. If F and G have densities f and g, respectively, then $F * G$ has a density $f * g$ given by

$$f * g(t) = \int_0^t g(t-x)f(x)dx \quad . \tag{2.2.6}$$

When $F = G$, $F * F$ is denoted by F_2. Similarly, we denote by F_n the n-fold convolution of F with itself, that is,

$$F_n = \underbrace{F * F * \dots * F}_{(n \text{ terms})} . \tag{2.2.7}$$

We have then

$$F_0(t) \equiv 1 \quad \text{for} \quad t \geq 0 \quad \text{and} \tag{2.2.8}$$

$$F_{n+1}(t) = F_n * F(t) = \int_0^t F_n(t-x)dF(x) , \quad \text{for } n = 1, 2, \dots \tag{2.2.9}$$

Therefore, from (2.2.7) and (2.2.8),

$$\begin{aligned} P\{N_t = n\} &= P\{N_t \geq n\} - P\{N_t \geq n+1\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} \\ &= F_n(t) - F_{n+1}(t) , \end{aligned} \tag{2.2.10}$$

which, using the notation $P\{S_n \leq t\} = F_{S_n}(t) = F_n(t)$, can be proved as follows:

Since $\{N_t = \sup(n: s_n \leq t)\}$ implies that $\{S_{N_t} \leq t\}$ and $\{S_{N_t+1} > t\}$,

$$P\{N_t = n\} = \int_0^t P\{S_{n+1} > t \mid S_n = s\}\, dP\{S_n \leq s\}$$

$$= \int_0^t P\{X_{n+1} > t - s\}\, dP\{S_n \leq s\}$$

$$= \int_0^t [1 - F(t-s)]\, dF_n(s)$$

$$= F_n(t) - \int_0^t F(t-s)\, dF_n(s)$$

$$= F_n(t) - \int_0^t F_n(t-s)\, dF(s) \,,$$

(using integration by parts),

$$= F_n(t) - F_{n+1}(t) \,.$$

The following theorem, the proof of which appears in Prabhu (1965), is useful for validating some of the steps below.

<u>Theorem II.B.1</u>: N_t is a well-defined random variable, with finite moments of all orders, that is,

a) $P\{N_t < \infty\} = 1$,

b) $E\{N_t\}^k < \infty$, for $k = 1, 2, \ldots$

The result of the following lemma is well-known, but a verification is given here for completeness.

<u>Lemma II.B.1</u>: The mean of the random variable N_t is given by

$$E\{N_t\} = \sum_{n=1}^{\infty} F_n(t) .$$

Proof:

$$E\{N_t\} = \sum_{n=0}^{\infty} n \cdot P\{N_t = n\} = \sum_{n=0}^{\infty} n[F_n(t) - F_{n+1}(t)], \text{ using Eq. (2.2.10),}$$

$$= [F_1(t) - F_2(t)] + 2[F_2(t) - F_3(t)] + \ldots + (k-1)[F_{k-1}(t) - F_k(t)]$$

$$+ k[F_k(t) - F_{k+1}(t)] + \ldots$$

$$= F_1(t) + F_2(t) + \ldots + F_k(t) + \ldots$$

$$= \sum_{n=1}^{\infty} F_n(t) . \qquad \text{Q.E.D.}$$

The mean value function $E\{N_t\}$, denoted by $m(t)$, is called the renewal function. From Theorem II.B.1, $E\{N_t\} < \infty$ for all t. Furthermore, the LaPlace-Stieltjes transform of a function uniquely determines the function. It will be shown that $m(t)$ can be determined by using the corresponding Laplace-Stieltjes transform. The Laplace-Stieltjes transform can often be more conveniently used to determine the asymptotic distribution of a convolution. Therefore, we shall consider so-called 'The Laplace Transform Convolution Theorem', which will be applied later to determine the asymptotic limit distributions of $\mathbb{P}_t$ and $D_{(t-\tau,t]}$, and to prove the well-known Blackwell's Renewal Theorem in Section C of this chapter. Following two definitions appear in Holl, Maple and Vinograde (1959).

<u>Definition II.B.1</u>: A function $F(t)$ is said to be of exponential order e^{bt} if corresponding to the constant b there exists a pair of positive constants t_0 and M such that for all t at which $F(t)$ is defined and $t > t_0$,

$$|e^{-bt} F(t)| \leq M, \tag{2.2.11}$$

<u>Definition II.B.2</u>: A function $F(t)$ is defined to be of class $\mathfrak{F}$ if for some constant b it is of exponential order e^{bt} and sectionally continuous.

The Laplace-Stieltjes (or just Laplace) transform ψ of a function φ is defined as

$$L\{\varphi(t)\} = \psi(s) = \int_0^\infty e^{-st} \varphi(t)\, dt . \tag{2.2.12}$$

Integrating by parts,

$$L\{\varphi'(t)\} = \int_0^\infty e^{-st} \varphi'(t)\, dt$$

$$= e^{-st}\,\varphi(t)\Big|_0^\infty + \int_0^\infty s\,e^{-st}\,\varphi(t)\,dt$$

$$= s\int_0^\infty e^{-st}\,\varphi(t)\,dt = S\cdot L\{\varphi(t)\}$$

$$\therefore\ L\{\varphi(t)\} = \frac{1}{s}\cdot L\{\varphi'(t)\}\ . \tag{2.2.13}$$

Using the Laplace transform notation, the important convolution theorem shall be stated without proof. Its proof appears in Holl, Maple and Vinograde (1959), Doetsch (1961), and Widder (1971).

<u>Theorem II.B.2</u>: If $F(t)$ and $G(t)$ are of class $\mathfrak{F}$, then

$$L\{\int_0^t G(t-x)\,F(x)\,dx\} = L\{F(t)\}\cdot L\{G(t)\},\quad \text{for}\quad s > b\ ,$$

where e^{bt} is the maximum of the exponential orders of $F(t)$ and $G(t)$.

Now, using Theorem II.B.2 and Lemma II.B.1, one may show how $m(t)$ and F mutually determine each other. That is,

$$L\{m(t)\} = L\{\sum_{n=1}^\infty F_n(t)\} = \sum_{n=1}^\infty L\{F_n(t)\}$$

$$= \sum_{n=1}^\infty (L\{F(t)\})^n$$

$$= \frac{L\{F(t)\}}{1 - L\{F(t)\}} \tag{2.2.14}$$

from which it follows that

$$L\{F(t)\} = \frac{L\{m(t)\}}{1 + L\{m(t)\}}.$$

Hence, Eq. (2.2.14) shows the one-to-one correspondence between $m(t)$ and F .

Corollary II.B.1: If $F(t)$ and $G(t)$ are of class $\mathfrak{F}$, then, for $s > b$ and $t \geq \tau$,

$$L\{\int_0^{t-\tau} G(t-x)F(x)dx\} = L\{F(t)\} \cdot L\{G(t)\} - L\{F(t)\} \cdot \int_0^{\tau} e^{-sy} G(y)dy , \tag{2.2.15}$$

where τ is a nonnegative constant.

Proof:

Define

$$I(A) = \iint_A e^{-s(x+y)} F(x) G(y) dx dy$$

$$= \int_0^k e^{-sx} F(x) dx \int_0^k e^{-sy} G(y) dy ,$$

such that the region A of integration is illustrated in Fig. II.1.

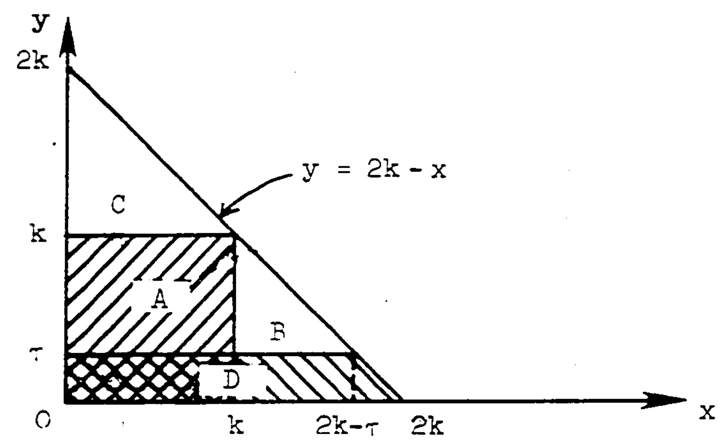

Figure II.1. Illustration of the domain of integration I(A).

Then

$$L\{F(t)\} \cdot L\{G(t)\} = \left(\int_0^\infty e^{-st} F(t)\, dt \right) \left(\int_0^\infty e^{-st} G(t)\, dt \right)$$

$$= \lim_{k \to \infty} \int_0^k e^{-sx} F(x)\, dx \int_0^k e^{-sy} G(y)\, dy$$

$$= \lim_{k \to \infty} I(A) .$$

Similarly,

$$L\{ \int_0^{t-\tau} G(t-x)\, F(x)\, dx\} = \lim_{k \to \infty} \int_0^{2k} e^{-st} \left(\int_0^{t-\tau} G(t-x)F(x)dx \right) dt ,$$

whose integral is equal to a double integral over the triangular region shown in Fig. II.2.

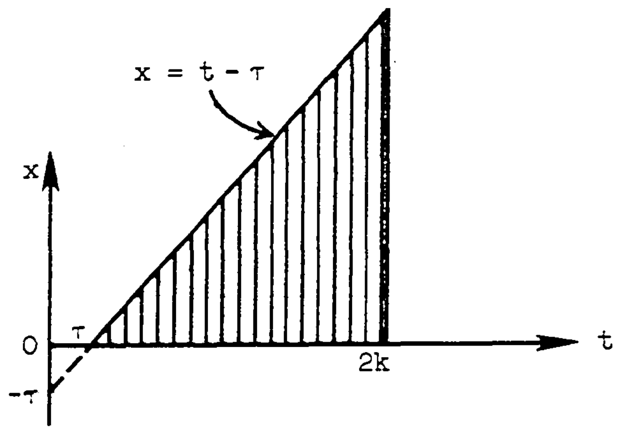

Figure II.2. Domain of integration I(R).

$$= \lim_{k \to \infty} \int_{\tau}^{2k} e^{-st} \left(\int_{0}^{t-\tau} G(t-x)\, F(x)\, dx \right) dt \,, \quad \text{for} \quad t \geq \tau$$

$$= \lim_{k \to \infty} \int_{0}^{2k-\tau} F(x)\, dx \int_{x+\tau}^{2k} e^{-st}\, G(t-x)\, dt$$

$$= \lim_{k \to \infty} \int_{0}^{2k-\tau} F(x)\, dx \int_{\tau}^{2k-x} e^{-s(x+y)}\, G(y)\, dy \,,$$

replaced $t-x$ by y ,

$$= \lim_{k \to \infty} I(R) \,,$$

where the region of integration R is composed of the three domains $A \cap \tilde{D}$, B and C ($\tilde{D}$ = the complement of D) in Fig. II.1.

However,

$$L\{\int_0^t G(t-x)\,F(x)\,dx\} = \lim_{k \to \infty} \int_0^{2k} e^{-st}\left(\int_0^t G(t-x)\,F(x)\,dx\right) dt$$

$$= \lim_{k \to \infty} \int_0^{2k} F(x)\,dx \int_0^{2k-x} e^{-s(x+y)}\,G(y)\,dy$$

$$= \lim_{k \to \infty} I(R') ,$$

where the region of integration R' is composed of the domains $A \cap \tilde{D}$, B, C and D.

Since

$$\lim_{k \to \infty} I(R') = L\{F(t)\} \cdot L\{G(t)\} ,$$

$$L\{\int_0^{t-\tau} G(t-x)\,F(x)\,dx\} = L\{F(t)\} \cdot L\{G(t)\} - \lim_{k \to \infty} I(D)$$

$$= L\{F(t)\} \cdot L\{G(t)\} - \lim_{k \to \infty} \int_0^{\tau} e^{-sy}\,G(y)\,dy$$

$$\cdot \int_0^{2k-y} e^{-sx}\,F(x)\,dx ,$$

where, given $0 \leq y \leq \tau$,

$$\lim_{k \to \infty} \left| \int_{2k-\tau}^{2k} e^{-sx} F(x)\, dx \right| \leq \lim_{k \to \infty} \int_{2k-\tau}^{2k} e^{-sx} |F(x)|\, dx = 0$$

for the convergence of $\int_0^\infty e^{-sx} |F(x)|\, dx$ for $s > b$,

$$= L\{F(t)\} \cdot L\{G(t)\} - \lim_{k \to \infty} \int_0^\tau e^{-sy} G(y)\, dy$$

$$\cdot \left[\int_0^{2k} e^{-sx} F(x)\, dx - \int_{2k-y}^{2k} e^{-sx} F(x)\, dx \right]$$

for $0 \leq y \leq \tau$,

$$= L\{F(t)\} \cdot L\{G(t)\} - \int_0^\tau e^{-sy} G(y)\, dy \{ \lim_{k \to \infty} \int_0^{2k} e^{-sx} F(x)\, dx \}$$

$$= L\{F(t)\} \cdot L\{G(t)\} - L\{F(t)\} \int_0^\tau e^{-sy} G(y)\, dy \quad .$$

Thus the proof is complete.

In order to determine the behavior of a distribution function as t tends to infinity, the Laplace transform of the distribution can often be used if the transform is known. This is illustrated by the next two theorems. The first theorem will be stated without proof. Its proof appears in Holl, Maple and Vinograde (1959) and Doetsch (1961).

<u>Theorem II.B.3</u>: If $F(t)$ is sectionally continuous with at most a finite number of discontinuities and of exponential order e^{bt}, and $F'(t)$ is also sectionally continuous, then

$$L\{F'(t)\} = s \cdot L\{F(t)\} - F(0^+) - \sum_{i=1}^{n} e^{-st_i} [F(t_i^+) - F(t_i^-)], \quad (s > b),$$

where $t_1, t_2, \ldots, t_n$ are the positive abscissas of the points of discontinuity of $F(t)$.

Professor B. Vinograde has helped us to prove the following theorem (see also Doetsch (1961)).

<u>Theorem II.B.4</u>: If $F(t)$ is of class $\mathfrak{J}$, and further if $F(t)$ has at most a finite number of discontinuities (at $t_1, t_2, \ldots, t_n$), and $F'(t)$ is of class $\mathfrak{J}$, then

$$\lim_{s \to 0^+} s \cdot L\{F(t)\} = \lim_{t \to \infty} F(t), \quad \text{for} \quad b < 0,$$

if either limit exists.

Proof: From Theorem II.B.3,

$$\lim_{s \to 0^+} s \cdot L\{F(t)\} = \lim_{s \to 0^+} [L\{F'(t)\} + F(0^+) + \sum_{i=1}^{n} e^{-st_i} \cdot \{F(t_i^+) - F(t_i^-)\}]$$

$$= \int_0^\infty \lim_{s \to 0^+} [e^{-st} F'(t)]dt + F(0^+) + \sum_{i=1}^{n} (F(t_i^+) - F(t_i^-)) ,$$

by the assumption of class $\mathfrak{F}$,

$$= \int_0^\infty F'(t)\,dt + F(0^+) + \sum_{i=1}^{n} (F(t_i^+) - F(t_i^-))$$

$$= \lim_{t \to \infty} F(t) ,$$

since

$$\int_0^t F'(x)\,dx = F(x)\Big|_{0^+}^{t_1^-} + F(x)\Big|_{t_1^+}^{t_2^-} + \dots + F(x)\Big|_{t_n^+}^{t}$$

$$= F(t) - F(0^+) - \sum_{j=1}^{n} (F(t_j^+) - F(t_j^-))$$

and thus

$$\int_0^\infty F'(x)\,dx = \lim_{t \to \infty} \int_0^t F'(x)\,dx$$

$$= \lim_{t \to \infty} F(t) - F(0^+) - \sum_{j=1}^{n} (F(t_j^+) - F(t_j^-)) .$$

The rest of this section cites some important renewal theorems, which will be used to study the distribution of procurement lead time demand $D_{(t-\tau,t]}$.

Let $F(t)$, $g(t)$, and $H(t)$ be functions defined for $t \geq 0$ satisfying the relation

$$g(t) = H(t) + \int_0^t g(t-x)\, d\, F(x)$$

where $F(t)$ and $H(t)$ are known functions, and $g(t)$ is an unknown function to be determined as the solution of the integral equation. The integral equation is so-called a renewal-type equation and its solution is given by the following theorem, the proof of which appears in Feller (1971), Prabhu (1965), and Ross (1970).

<u>Theorem II.B.5</u>: If

$$g(t) = H(t) + \int_0^t g(t-x)\, d\, F(x) \,, \quad (t \geq 0) \,,$$

then

$$g(t) = H(t) + \int_0^t H(t-x)\, d\, m(x) \,, \text{ where } m(x) = \sum_{n=1}^{\infty} F_n(x) \,.$$

As is pointed out in Parzen (1962) and Ross (1970), if the first demand (renewal) occurs at time x, $x \leq t$, then from this time point on the renewal process starts over again, and thus the expected number of renewals in $(0, t]$ is one plus the expected number to arrive in a time $t-x$ from the beginning of an equivalent renewal process. Therefore,

$$E\{N_t \mid X_1 = x\} = \begin{cases} 1 + m(t-x) , & \text{if} \quad x \leq t \\ 0 , & \text{if} \quad x > t \end{cases} .$$

Thus, the mean value function $m(t)$ of the renewal counting process $\{N_t ;\ t \in T\}$ is also stated in the form of a renewal-type equation;

$$m(t) = E\{N_t\} = \sum_{n=1}^{\infty} F_n(t) = \int_0^{\infty} E\{N_t \mid X_1 = x\}\, d\,F(x)$$

$$= \int_0^{t} (1 + m(t-x))\, d\,F(x)$$

$$= F(t) + \int_0^{t} m(t-x)\, d\,F(x) \qquad (2.2.16)$$

$$= F(t) + \int_0^{t} F(t-x)\, d\,m(x) \text{ , using integration by part.}$$

A nonnegative random variable X is said to be lattice if there exists $d \geq 0$ such that $\sum_{m=0}^{\infty} P\{X = md\} = 1$. Since, according to Feller (1971), Parzen (1962) and Ross (1970), a lattice random variable X is defined to be a discrete random variable with the property that all values x which X can assume with positive probability are of the form $x = md$, for some real number d , and integer m , an integer-valued random variable is a lattice random variable. Feller defines the distribution of such a random variable to be arithmetic. We now state without proof the Key Renewal Theorem which will be used later to determine the asymptotic limit distribution of $D_{(t-\tau,t]}$ as $t \rightarrow \infty$. It has been proved by Smith (1958) and Takacs (1958).

Theorem II.B.6 (Key Renewal Theorem): If the inter-arrival time X has finite mean μ and the distribution F is not arithmetic, and $H(t)$ is any function satisfying the conditions

a) $H(t) \geq 0$ for all $t \geq 0$,

b) $\int_0^\infty H(t)\, dt < \infty$,

c) $H(t)$ is nonincreasing ,

then it is true that

$$\lim_{t \rightarrow \infty} \int_0^t H(t-x)\, d\, m(x) = \frac{1}{\mu} \int_0^\infty H(t)\, dt .$$

C. Joint Distribution of Inventory Position and Incremental Demand under the $<Q, r>$ Model

In this section, under the $<Q, r>$ policy, we shall first find the marginal distribution functions of $\{IP_t;\ t \in T\}$, $\{D_{(t-\tau,t]};\ t \in T\}$ and the residual waiting time $\{Z_{t-\tau};\ t \in T\}$, and then prove that $IP_{t-\tau}$ and $D_{(t-\tau,t]}$ are mutually independent of each other.

An inventory position IP_t at time t totally depends upon the demand process $\{D_t;\ t \in T\}$. If an inventory system is started with

$IP_0 = r+i$ $(i = 1, 2, \ldots, Q)$ at time $t = 0$, then $IP_{t-\tau} = r+j$ $(j = 1, 2, \ldots, Q)$ at time $t-\tau > 0$ can be reached after the $(i-j)^+$ or $\{i + (m-1)Q + (Q-j);\ m = 1, 2, \ldots\}$ demand materialization by time $t-\tau$, where m denotes the total number of order placements by time $t-\tau$ and

$$(i - j)^+ = \max\{0, i-j\} . \tag{2.3.1}$$

Suppose now that we consider the sequence of events consisting of the times at which an order in the amount of Q is placed and received in the constant lead time τ . Defining Y_k to be the time elapsed between the $(k-1)^{st}$ and k^{th} orders, the sequence of random variables $\{Y_k;\ k = 1, 2, \ldots\}$ forms a modified renewal process in which the distribution functions are given by

$$P\{Y_1 \le y_1\} = P\{S_i \le y_1\} \equiv F_i(y_1) = P\{N_{y_1} \ge i\} , \tag{2.3.2}$$

where i is the initial stock over the reorder point r , and likewise,

$$P\{Y_k \le y_k\} = P\{S_Q \le y_k\} = P\{N_{y_k} \ge Q\} = F_Q(y_k) , \tag{2.3.3}$$

$$\text{for } k = 2, 3, \ldots,$$

since

$$\{Y_k \le y_k\} \iff \{(S_{i+(k-1)Q} - S_{i+(k-2)Q}) \le y_k\}$$

$$\Longleftrightarrow \quad \{S_Q \leq y_k\} \quad \text{for} \quad k = 2, 3, \ldots$$

Thus, a new renewal process $\{W_m;\ m = 0, 1, 2, \ldots\}$ is defined such that

$$W_0 = Y_0 = 0$$
$$W_m = \sum_{k=1}^{m} Y_k = S_{i+(m-1)Q}\,, \qquad m = 1, 2, 3, \ldots \tag{2.3.4}$$

where 'm = 0' means that no order is placed yet. Let $(t - \tau - \theta)$ and m be, respectively, particular values of the time T and the serial number M of the last order placed no later than $t - \tau$. If we assume that $IP_{t-\tau} = r + j$ $(j = 1, 2, \ldots, Q)$ at time $t - \tau$, then we see that $(Q - j)$ demands are further needed in the time interval $(t - \tau - \theta, t - \tau]$, for $\theta \geq 0$, since the inventory position at time $t - \tau - \theta$ is $r + Q$ immediately after the m^{th} order is placed at time $t - \tau - \theta$.

<u>Theorem II.C.1</u>: For the continuous-review $\langle Q, r\rangle$ inventory system with backorders allowed, constant lead time $\tau \geq 0$, iid customer inter-arrival times with finite mean, units demanded one at a time, and with $IP_0 = r + i$ $(i = 1, 2, \ldots, Q)$,

$$P\{IP_{t-\tau} = r + j\} = P\{N_{t-\tau} = (i-j)\}^{+} + \sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{N_\theta = Q - j\}\, d\,P\{W_m \leq t - \tau - \theta\},$$

$$\text{for} \quad j = 1, 2, \ldots, Q\,,$$

where $P\{N_{t-\tau} = (i-j)\}^{+} = 0$, if $i < j$.

Proof:

Denote by $\phi_m\{T \leq t-\tau-\theta\}$ the probability that $M = m$ and $T \leq t-\tau-\theta$ so that $\phi_m\{T \leq t-\tau-\theta\} = P\{W_m \leq t-\tau-\theta\}$.

Since the inventory position $IP_{t-\tau} = r+j$ $(j = 1, 2, \ldots, Q)$ can be reached after the demand materialization $D_{(0,t-\tau]}$ such that

$$D_{(0,t-\tau]} = N_{t-\tau} = (i-j)^{+} , \quad \text{for} \quad m = 0$$

$$D_{(0,t-\tau]} = N_{t-\tau} = N_{t-\tau-\theta} + (N_{t-\tau} - N_{t-\tau-\theta}) = \{i + (m-1)Q\} + (Q-j), \quad \text{for} \quad m = 1, 2, \ldots ,$$

$$P\{IP_{t-\tau} = r+j\} = \sum_{m=0}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{IP_{t-\tau} = r+j \mid M=m, \; T=t-\tau-\theta\} \cdot d\, \phi_m\{T \leq t-\tau-\theta\}$$

$$= P\{N_{t-\tau} = (i-j)\}^{+} + \sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{N_{t-\tau} - N_{t-\tau-\theta} = Q-j \mid M=m, \; T=t-\tau-\theta\} \, d\, P\{W_m \leq t-\tau-\theta\}$$

$$= P\{N_{t-\tau} = (i-j)\}^{+} + \sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{N_{\theta} = Q-j\} \, d\, P\{W_m \leq t-\tau-\theta\} .$$

Let $Z_{t-\tau}$ be the time from $t-\tau$ until the first demand subsequent to $t-\tau$, that is,

$$Z_{t-\tau} = S_{N_{t-\tau}+1} - (t-\tau) , \tag{2.3.5}$$

where $S_{N_{t-\tau}} \leq t-\tau < S_{N_{t-\tau}+1}$.

The variable $Z_{t-\tau}$ will be the residual or excess waiting time at epoch $t-\tau$. The distribution function of $Z_{t-\tau}$ can be determined by use of the renewal equation for $m(t)$.

<u>Theorem II.C.2</u>: For the inventory model of Theorem II.C.1,

$$P\{Z_{t-\tau} \leq z\} = F(t-\tau+z) - \int_0^{t-\tau} [1 - F(t-\tau+z-\xi)]\, d\,m(\xi)$$

$$= \int_{t-\tau}^{t-\tau+z} [1 - F(t-\tau+z-\xi)]\, d\,m(\xi) , \quad \text{for} \quad z > 0 .$$

Proof:

From Eqs. (2.2.1) and (2.2.2), $S_{N_{t-\tau}} \leq t-\tau$.

$$P\{Z_{t-\tau} \leq z\} = P\{0 < S_{N_{t-\tau}+1} - (t-\tau) \leq z\} = P\{t-\tau < S_{N_{t-\tau}+1} \leq t-\tau+z\}$$

$$= P\{t-\tau - X_1 \le t-\tau+z\} + \sum_{n=1}^{\infty} \int_0^{t-\tau} P\{t-\tau < S_{n+1} \le t-\tau+z \mid S_n = \xi\} \cdot d\, P\{S_n \le \xi\}$$

$$= F(t-\tau+z) - F(t-\tau) + \int_0^{t-\tau} P\{t-\tau-\xi < X_{n+1} \le t-\tau+z-\xi\} \cdot \sum_{n=1}^{\infty} d\, P\{S_n \le \xi\}$$

$$= F(t-\tau+z) - F(t-\tau) + \int_0^{t-\tau} [F(t-\tau+z-\xi) - F(t-\tau-\xi)]\, d\, m(\xi) ,$$

(from Lemma 2.2.1) ,

$$= F(t-\tau+z) + \int_0^{t-\tau} F(t-\tau+z-\xi)\, d\, m(\xi) - [F(t-\tau) + \int_0^{t-\tau} F(t-\tau-\xi) \cdot d\, m(\xi)]$$

$$= F(t-\tau+z) + \int_0^{t-\tau} F(t-\tau+z-\xi)\, d\, m(\xi) - m(t-\tau) ,$$

from Eq. (2.2.16) ,

$$= F(t-\tau+z) - \int_0^{t-\tau} [1 - F(t-\tau+z-\xi)]\, d\, m(\xi) \qquad (2.3.6)$$

$$= \int_{t-\tau}^{t-\tau+z} [1 - F(t-\tau+z-\xi)]\, d\, m(\xi) , \qquad (2.3.7)$$

since

$$F(t-\tau+z) = m(t-\tau+z) - \int_0^{t-\tau+z} F(t-\tau+z-\xi)\, d\, m(\xi), \text{ from Eq. (2.2.16).}$$

Let $t-\tau+z$ be the time point at which the first demand occurs after time $t-\tau$. The random variable $Z_{t-\tau}$ may have a different distribution from those of X_i's. The distribution of $D_{(t-\tau,t]}$ is determined in the next Theorem II.C.3 by partitioning in accordance with the time $t-\tau+z$ at which the first demand occurs after time $t-\tau$ and the time interval $(t-\tau+z, t]$ in which $k-1$ demands occur.

<u>Theorem II.C.3</u>: Under the assumptions made in Theorem II.C.1,

$$P\{D_{(t-\tau,t]} = k\} = \begin{cases} \int_0^{\tau} P\{N_{\tau-z} = k-1\}\; d\; P\{Z_{t-\tau} \le z\}, & \text{for } k = 1, 2, \ldots \\ P\{Z_{t-\tau} > \tau\}, & \text{for } k = 0 \end{cases} .$$

Proof:

For $k = 0$,

$$P\{D_{(t-\tau,t]} = 0\} = P\{N_t - N_{t-\tau} = 0\}$$

$$= P\{Z_{t-\tau} > \tau\} .$$

For $k \geq 1$,

$$P\{D_{(t-\tau,t]} = k\} = P\{N_t - N_{t-\tau} = k\}$$

$$= \int_0^\infty P\{N_t - N_{t-\tau} = k \mid Z_{t-\tau} = z\} \, d\, P\{Z_{t-\tau} \le z\}$$

$$= \int_0^\tau P\{N_{\tau-z} = k-1\} \, d\, P\{Z_{t-\tau} \le z\}$$

$$= \int_0^\tau [F_{k-1}(\tau - z) - F_k(\tau - z)] \, d\, P\{Z_{t-\tau} \le z\} .$$

Alternative Proof:

Given $N_{t-\tau} = n$ and $N_t = n+k$ $(n, k = 0, 1, 2, \ldots)$, $S_{N_t} - (t-\tau)$ is formed as follows:

$$S_{N_t} - (t-\tau) = Z_{t-\tau} + Z_{n+2} + X_{n+3} + \cdots + X_{n+k} .$$

Let

$$\tilde{S}_k = S_{N_t} - (t-\tau), \quad P\{\tilde{S}_k \le \tau\} = \tilde{F}_k(\tau), \text{ and } G(z) = P\{Z_{t-\tau} \le z\} .$$

Then,

$$P\{D_{(t-\tau,t]} = k\} = P\{N_t - N_{t-\tau} = k\} \overset{\text{say}}{=} P\{\tilde{N}_\tau = k\}$$

$$= \tilde{F}_k(\tau) - \tilde{F}_{k+1}(\tau) ,$$

where

$$\tilde{F}_k(\tau) = P\{\tilde{S}_k \le \tau\} = F_{k-1} * G(\tau)$$

$$= \int_0^{\tau} F_{k-1}(\tau - z)\, d\, G(z)$$

$$= \int_0^{\tau} P\{N_{\tau-z} \geq k-1\}\, d\, P\{Z_{t-\tau} \leq z\} .$$

Therefore, for $k = 0$,

$$P\{D_{(t-\tau,t]} = 0\} = \tilde{F}_0(\tau) - \tilde{F}_1(\tau)$$

$$= 1 - G(\tau) , \quad \text{since } F_0(x) \equiv 1 \text{ for } x \geq 0 \text{ and } \tilde{F}_0(\tau) = 1$$

$$= 1 - P\{Z_{t-\tau} \leq \tau\}$$

$$= P\{Z_{t-\tau} > \tau\} ,$$

and for $k \geq 1$,

$$P\{D_{(t-\tau,t]} = k\} = F_{k-1} * G(\tau) - F_k * G(\tau)$$

$$= \int_0^{\tau} [F_{k-1}(\tau - z) - F_k(\tau - z)]\, d\, G(z)$$

$$= \int_0^{\tau} P\{N_{\tau-z} = k-1\}\, d\, P\{Z_{t-\tau} \leq z\} .$$

$\therefore$ The proof is complete.

The expectation of $D_{(t-\tau,t]}$ is formed as follows:

$$E\{D_{(t-\tau,t]}\} = \sum_{k=0}^{\infty} k\ P\{D_{(t-\tau,t]} = k\}$$

$$= \sum_{k=1}^{\infty} k \int_0^{\tau} [F_{k-1}(\tau - z) - F_k(\tau - z)]\ d\ P\{Z_{t-\tau} \le z\}\ ,$$

(from Theorem II.C.3) ,

$$= \int_0^{\tau} [\sum_{k=1}^{\infty} k\ \{F_{k-1}(\tau - z) - F_k(\tau - z)\}]\ d\ P\{Z_{t-\tau} \le z\}$$

$$= \int_0^{\tau} [\{F_0(\tau - z) - F_1(\tau - z)\} + 2\{F_1(\tau - z) - F_2(\tau - z)\}$$

$$+ 3\{F_2(\tau - z) - F_3(\tau - z)\} + \cdots]\ d\ P\{Z_{t-\tau} \le z\}$$

$$= \int_0^{\tau} (\sum_{k=0}^{\infty} F_k(\tau - z))\ d\ P\{Z_{t-\tau} \le z\}$$

$$= \int_0^{\tau} [1 + \sum_{k=1}^{\infty} F_k(\tau - z)]\ d\ P\{Z_{t-\tau} \le z\}\ , \qquad \text{(A)}$$

since $F_0(x) \equiv 1$ for $x \ge 0$,

$$= P\{Z_{t-\tau} \le \tau\} + \sum_{k=1}^{\infty} \int_0^{\tau} F_k(\tau - z)\ d\ P\{Z_{t-\tau} \le z\}$$

$$= G(\tau) + \sum_{k=1}^{\infty} F_k * G(\tau)\ , \quad \text{where} \quad G(z) = P\{Z_{t-\tau} \le z\}$$

$$= G(\tau) + \int_0^{\tau} m(\tau - z)\, d\, G(z)\ , \quad \text{(directly from (A))}\ ,$$

where $m(\tau - z) = \sum_{k=1}^{\infty} F_k(\tau - z)$.

As we saw in the proof of Theorem II.C.1, $P\{IP_{t-\tau} = x\}$ is a function of $P\{N_{t-\tau} = y\}$ which means that the inventory position $IP_{t-\tau}$ is determined by $N_{t-\tau}$. However, we shall prove that given $IP_0 = r + i$ $(i = 1, 2, \ldots, Q)$ at time $t = 0$, for any distribution of the inter-arrival times between demands the distribution of $IP_{t-\tau}$ is independent of that of $D_{(t-\tau,t]}$, even though $D_{(t-\tau,t]} = N_t - N_{t-\tau}$.

<u>Theorem II.C.4</u>: Under the assumptions made in Theorem II.C.1,

$$P\{IP_{t-\tau} = r + j,\ D_{(t-\tau,t]} = k\} = P\{IP_{t-\tau} = r + j\}\ P\{D_{(t-\tau,t]} = k\}\ ,$$

$$\text{for } j = 1, 2, \ldots, Q \text{ and } k = 0, 1, 2, \ldots$$

Proof:

As is done for the proof of Theorem II.C.1,

$$P = \{IP_{t-\tau} = r + j,\ D_{(t-\tau,t]} = k\} = \sum_{m=0}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{IP_{t-\tau} = r + j,$$

$$D_{(t-\tau,t]} = k \mid M = m,\ T = t - \tau - \theta\}\ d\phi_m\{T \le t - \tau - \theta\}$$

$$= \left[\begin{array}{l} P\{N_{t-\tau} = (i-j),\ N_t - N_{t-\tau} = k\}^+ \ + \\[2ex] \displaystyle\sum_{m=1}^{\infty} \int_{\theta=0}^{\theta-t-\tau} P\{N_{t-\tau} - N_{t-\tau-\theta} = Q-j,\ N_t - N_{t-\tau} = k \mid M=m,\ T=t-\tau-\theta\} \\[2ex] \qquad\qquad\qquad\qquad \cdot\, d\, P\{W_m \leq t-\tau-\theta\} \end{array} \right]$$

where $P\{N_{t-\tau} = (i-j),\ N_t - N_{t-\tau} = k\}^+ = 0$ if $i < j$

$$= \left[\begin{array}{l} P\{N_{t-\tau} = (i-j),\ N_t - N_{t-\tau} = k\}^+ \ + \\[2ex] \displaystyle\sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{N_{t-\tau} - N_{t-\tau-\theta} = Q-j,\ N_t - N_{t-\tau} = k\}\, d\, P\{W_m \leq t-\tau-\theta\} \end{array} \right]$$

$$= \left[\begin{array}{l} \displaystyle\int_{z=0}^{z=\tau} P\{N_{t-\tau} = (i-j),\ N_t - N_{t-\tau} = k \mid Z_{t-\tau} = z\}^+\, d\, P\{Z_{t-\tau} \leq z\} \ + \\[2ex] \displaystyle\sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} \int_{z=0}^{z=\tau} P\{N_{t-\tau} - N_{t-\tau-\theta} = Q-j,\ N_t - N_{t-\tau} = k \mid Z_{t-\tau} = z\} \\[2ex] \qquad\qquad\qquad\qquad \cdot\, d\, P\{Z_{t-\tau} \leq z\}\ d\, P\{W_m \leq t-\tau-\theta\} \end{array} \right]$$

$$= \left[\begin{array}{l} \int_{z=0}^{z=\tau} P\{N_{t-\tau} = (i-j)\,,\ N_{\tau-z} = k-1\}^{+}\ d\,P\{Z_{t-\tau} \le z\} \ + \\[2ex] \sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} \int_{z=0}^{z=\tau} P\{N_{\theta} = Q-j,\ N_{\tau-z} = k-1\}\, d\,P\{Z_{t-\tau} \le z\}\, d\,P\{W_m \le t-\tau-\theta\} \end{array} \right]$$

$$= \left[\begin{array}{l} \int_{z=0}^{z=\tau} P\{N_{t-\tau} = (i-j)\}^{+}\ P\{N_{\tau-z} = k-1\}\ d\,P\{Z_{t-\tau} \le z\} \ + \\[2ex] \sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} \int_{z=0}^{z=\tau} [P\{N_{\theta} = Q-j\}\ P\{N_{\tau-z} = k-1\}]\, d\,P\{Z_{t-\tau} \le z\}\, d\,P\{W_m \le t-\tau-\theta\} \end{array} \right]$$

$$= \left[\begin{array}{l} P\{N_{t-\tau} = (i-j)\}^{+} \int_{z=0}^{z=\tau} P\{N_{\tau-z} = k-1\}\ d\,P\{Z_{t-\tau} \le z\} \ + \\[2ex] \sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{N_{\theta} = Q-j\}\, d\,P\{W_m \le t-\tau-\theta\} \int_{z=0}^{z=\tau} P\{N_{\tau-z} = k-1\}\, d\,P\{Z_{t-\tau} \le z\} \end{array} \right]$$

$$= \left[P\{N_{t-\tau} = (i-j)\}^{+} + \sum_{m=1}^{\infty} \int_{\theta=0}^{\theta=t-\tau} P\{N_{\theta} = Q-j\}\, d\,P\{W_m \le t-\tau-\theta\} \right]$$

$$\cdot \int_{z=0}^{z=\tau} P\{N_{\tau-z} = k-1\}\ d\,P\{Z_{t-\tau} \le z\}$$

$= P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = k\}$, from Theorems II.C.1 and II.C.3.

$\therefore$ The proof is complete.

D. Limit Distributions

In this section, we shall find the limit distributions of $IP_{t-\tau}$ and $D_{(t-\tau,t]}$, and of their joint distribution, as $t \longrightarrow \infty$. The Laplace-Stieltjes transforms of those distribution functions can often be used as is done below to determine such limit distributions. However, the limit distribution of $D_{(t-\tau,t]}$ is more easily found by applying the Key Renewal Theorem.

It has been shown in Galliher, Morse and Simond (1958), and Hadley and Whitin (1963) that under the $<Q, r>$ policy with one-at-a-time demand process, when the inter-arrival times X_i in a continuous-review inventory system are independent, identically distributed and have negative exponential distribution, the limiting distribution of the inventory position is uniform on the set $\{r+1, r+2, \ldots, r+Q\}$. Simon (1968) showed the same uniform distribution on the set $\{r+1, r+2, \ldots, r+Q\}$ with the assumption of arbitrary inter-arrival time distributions under a continuous-review $< nQ, r >$ replenishment policy, under which an amount nQ is ordered at the time of an inventory review, where n denotes the nonnegative integer which will put the inventory position on the set $\{r+1, r+2, \ldots, r+Q\}$. His result holds even when the demand quantity is random and the procurement lead time for orders placed are random variables with arbitrary distributions. With the restriction that units be demanded one at a time, Sivazlian (1974) has considered the direct Laplace-Stieltjes transform approach to determine the limiting distribution of the inventory position and obtained the same result, namely, uniformity on the set $\{r+1, r+2, \ldots, r+Q\}$ regardless of the distribution of the inter-arrival times between demands whenever the system operates under the $\langle Q, r\rangle$ policy. Richards (1975) seems to suggest that the result of Sivazlian is a special case of the result given by Simon, and considered the case of random demand size in which the limiting distribution is shown not uniform under the $\langle Q, r\rangle$ policy.

Under the assumption that the lead time τ be constant, units are demanded one at a time, unfilled demands be completely backordered and the $\langle Q, r\rangle$ policy be used, we shall consider the direct Laplace-Stieltjes transform approach and/or the application of the Key Renewal Theorem to the inventory system to determine the limiting distributions of the inventory position processes, of the lead time demand processes, and of the joint distribution of them.

<u>Theorem II.D.1</u>: Under the assumptions made in Theorem II.C.1,

$$H_1(j) = \lim_{t \to \infty} P\{IP_{t-\tau} = r+j\} = \frac{1}{Q} \quad (j = 1, 2, \ldots, Q)$$

if and only if all demands are of unit size.

Proof:

Let

$$\hat{F}(s) = L\,[P\{IP_{t-\tau} = r+j\}] \quad \text{for} \quad i,j = 1, 2, \ldots, Q \;.$$

Then,

$$\hat{F}(s) = \int_0^\infty e^{-st} P\{IP_{t-\tau} = r + j\} dt$$

$$= \int_0^\infty e^{-st} [P\{N_{t-\tau} = (i-j)\}^+ + \sum_{m=1}^{\infty} \int_{\Theta=0}^{\Theta=t-\tau} P\{N_\Theta = Q - j\}$$

$$\cdot d P\{W_m \leq t - \tau - \theta\}] dt, \text{ from Theorem II.C.1,}$$

$$= \int_0^\infty e^{-st} [F_{(i-j)^+}(t - \tau - F_{(i-j)^+ + 1}(t - \tau)$$

$$+ \sum_{m=1}^{\infty} \int_{\Theta=0}^{\Theta=t-\tau} \{F_{Q-j}(\Theta) - F_{Q-j+1}(\Theta)\} f_{i+(m-1)Q}(t - \tau - \Theta) d\Theta] dt ,$$

from Eq. (2.3.4), where

$$f_{i+(m-1)Q}(\Theta) = \frac{d}{d\Theta} F_{it(m-1)Q}(\Theta) = \frac{d}{d\Theta} P\{W_m \leq t - \tau - \Theta\} ,$$

$$= e^{-st} \left\{ \int_0^\infty e^{-su} [F_{(i-j)^+}(u) - F_{(i-j)^+ + 1}(u)] du \right.$$

$$\left. + \sum_{m=1}^{\infty} \int_0^\infty e^{-su} [\int_{\Theta=0}^{\Theta=u} (F_{Q-j}(\Theta) - F_{Q-j+1}(\Theta)) f_{i+(m-1)Q}(u-\Theta) d\Theta] du \right\},$$

replacing $t - \tau$ by u .

Then, following the same procedure shown in Sivazlian (1974) and applying Theorem II.B.2 and Theorem II.B.4, it is verified that the inventory position is uniformly distributed on the set $\{r+1, r+2, \ldots, r+Q\}$ and is independent of the distribution of the interarrival times between demands. It is not affected by the initial inventory position either.

In order to determine the limiting distribution of the lead time demand $D_{(t-\tau,t]}$, it is necessary to know the limiting distribution of the residual waiting time $Z_{t-\tau}$ at time $t-\tau$. This can be done through using the Key Renewal Theorem II.B.6. We know that the mean interarrival time is expressed as follows:

$$\begin{aligned}\mu = E[X] &= \int_0^\infty x \, d\, F(x) \\ &= \int_0^\infty [1 - F(x)]\, dx \,, \quad \text{taking the integration by part.}\end{aligned} \tag{2.4.1}$$

<u>Theorem II.D.2</u>: Under the assumptions made in Theorem II.C.1,

$$\lim_{t \to \infty} P\{Z_{t-\tau} \leq z\} = \frac{1}{\mu} \int_0^z [1 - F(x)]\, dx \,, \quad \text{for} \quad z \geq 0 \,.$$

Proof:

From Theorem II.C.2,

$$P\{Z_{t-\tau} \leq z\} = F(t-\tau+z) - \int_0^{t-\tau} [1 - F(t-\tau+z-\xi)]\, dm(\xi) .$$

$$\therefore \quad P\{Z_{t-\tau} > z\} = 1 - F(t-\tau+z) + \int_0^{t-\tau} [1 - F(t-\tau+z-\xi)]\, dm(\xi)$$

$$= h(t-\tau) + \int_0^{t-\tau} h(t-\tau-\xi)\, dm(\xi) ,$$

letting $h(t) = 1 - F(t+z)$

$$= h(y) + \int_0^{y} h(y-\xi)\, dm(\xi) , \quad \text{replacing } t-\tau \text{ by } y .$$

$$\therefore \quad \lim_{t \to \infty} P\{Z_{t-\tau} > z\} = \lim_{y \to \infty} [h(y) + \int_0^{y} h(y-\xi)\, dm(\xi)]$$

$$= 0 + \lim_{y \to \infty} \int_0^{y} h(y-\xi)\, dm(\xi) ,$$

since $\lim_{t \to \infty} F(t) = 1$,

$$= \frac{1}{\mu} \int_0^{\infty} h(y)\, dy ; \quad \text{from the Key Renewal Theorem II.B.6,}$$

$$= \frac{1}{\mu} \int_{\tau}^{\infty} [1 - F(t-\tau+z)]\, dt$$

$$= \frac{1}{\mu} \int_{z}^{\infty} [1 - F(x)]\, dx, \quad \text{replacing } t-\tau+z \text{ by } x,$$

$$\therefore \quad \lim_{t \to \infty} P\{Z_{t-\tau} \le z\} = 1 - \lim_{t \to \infty} P\{Z_{t-\tau} > z\}$$

$$= \frac{1}{\mu} \int_0^z [1 - F(x)]\, dx \, .$$

$\therefore$ The proof is complete.

Now, we can determine the limit distribution of $D_{(t-\tau,t]}$ using the above theorem or directly forming the Laplace-Stieltjes transform.

<u>Theorem II.D.3</u>: Under the assumptions made in Theorem II.C.1 ,

$$H_2(k) = \lim_{t \to \infty} P\{D_{(t-\tau,t]} = k)$$

$$= \begin{cases} \dfrac{\int_0^\tau F_{k-1}(y)\, dy - 2\int_0^\tau F_k(y)\, dy + \int_0^\tau F_{k+1}(y)\, dy}{\mu} & , \text{ for } k = 1,2,\ldots \\[2ex] 1 - \dfrac{1}{\mu} \int_0^\tau [1 - F(x)]\, dx & , \text{ for } k = 0 \, . \end{cases}$$

Proof:

For $k = 0$, from Theorem II.C.3,

$$\lim_{t \to \infty} P\{D_{(t-\tau,t]} = 0\} = \lim_{t \to \infty} P\{Z_{t-\tau} > \tau\}$$

$$= 1 - \lim_{t \to \infty} P\{Z_{t-\tau} \leq \tau\}$$

$$= 1 - \frac{1}{\mu} \int_0^\tau [1 - F(x)]\, dx \,, \quad \text{from Theorem II.D.2.}$$

For $k \geq 1$, from Theorem II.C.3,

$$\lim_{t \to \infty} P\{D_{(t-\tau,t]} = k\}$$

$$= \lim_{t \to \infty} P\{N_t - N_{t-\tau} = k\}$$

$$= \lim_{t \to \infty} \int_0^\tau P\{N_{\tau-z} = k-1\}\; d\, P\{Z_{t-\tau} \leq z\}$$

$$= \int_0^\tau P\{N_{\tau-z} = k-1\}\; d\, [\lim_{t \to \infty} P\{Z_{t-\tau} \leq z\}, \quad \text{from Helly-Bray Lemma,}$$

$$= \int_0^\tau P\{N_{\tau-z} = k-1\}\; \frac{1}{\mu}\, [1 - F(z)]\, dz$$

$$= \int_0^\tau [F_{k-1}(\tau - z) - F_k(\tau - z)]\; \frac{1}{\mu}\, [1 - F(z)]\, dz \,, \quad \text{from Theorem II.D.2,}$$

$$= \frac{1}{\mu} \int_0^\tau [F_{k-1}(\tau - z) - F_k(\tau - z)] dz - \frac{1}{\mu} \int_0^\tau [F_{k-1}(\tau - z) - F_k(\tau - z)] F(z)\, dz$$

$$= \frac{1}{\mu} \int_0^\tau [F_{k-1}(y) - F_k(y)]dy - \frac{1}{\mu} \int_0^\tau [F_{k-1}(\tau-z) - F_k(\tau-z)]\, F(z)\, dz \ ,$$

replacing $\tau - z$ by y

$$= \frac{\int_0^\tau F_{k-1}(y)dy - \int_0^\tau F_k(y)dy - \int_0^\tau F_{k-1}(\tau-z)\, F(z)dz + \int_0^\tau F_k(\tau-z)F(z)dz}{\mu}$$

$$= \frac{\int_0^\tau F_{k-1}(y)\, dy - 2\int_0^\tau F_k(y)\, dy + \int_0^\tau F_{k+1}(y)\, dy}{\mu} \ ,$$

since

$$\int_0^\tau F_n(y)\, dy = \int_0^\tau F_{n-1}(\tau - x)\, F(x)\, dx \qquad (2.4.2)$$

and its proof is as follows:

Define

$$\int_0^\tau F_n(y)\, dy = \overline{F_n(\tau)} \ .$$

Then, since $L\{F'(t)\} = L\{f(t)\} = s \cdot L\{F(t)\}$ from (2.2.13) ,

$$L\{ \int_0^\tau F_n(y)dy\} = L\{\overline{F_n(\tau)}\} = \frac{1}{s} \cdot L\{F_n(\tau)\} = \frac{1}{s^2} (\hat{f}(s))^n, \qquad (2.4.3)$$

where $\hat{f}(s) = \int_0^\infty e^{-st} \, d\, F(t)$.

And, from Theorem II.B.2,

$$L\{ \int_0^\tau F_{n-1}(\tau - x)\, F(x)\, dx\} = L\{F_{n-1}(\tau)\ L\{F(\tau)\}$$

$$= \{ \frac{1}{s} (\hat{f}(s))^{n-1} \} \cdot \{ \frac{1}{s} \hat{f}(s)\}$$

$$= \frac{1}{s^2} (\hat{f}(s))^n . \qquad (2.4.4)$$

Therefore, Eqs. (2.4.3) and (2.4.4) show that Eq. (2.4.2) holds.

∴ The proof is complete.

Remark:

$$L[P\{D_{(t-\tau,t]} = k\}] = L[P\{N_t - N_{t-\tau} = k\}]$$

$$= L[\int_0^\tau P\{N_{\tau-z} = k-1\}\, d\, P\{Z_{t-\tau} \leq z\} ,$$

from Theorem II.C.3 ,

$$= L[\int_0^\tau \{F_{k-1}(\tau-z) - F_k(\tau-z)\}$$

$$\cdot \{f(t-\tau+z) + \sum_{n=1}^{\infty} \int_0^{t-\tau} f(t-\tau+z-\xi)\, d\, F_n(\xi)\} dz] ,$$

from Theorem II.C.2,

$$= L[\int_{t-\tau}^{t} \{F_{k-1}(t-x) - F_k(t-x)\}\, dF(x) + \sum_{n=1}^{\infty} \int_0^{t-\tau} (\int_0^{z=\tau} \{F_{k-1}(t-z)$$

$$- F_k(t-x)\}\, dF(t-\tau+z-\xi))\, dF_n(\xi)] \ ,$$

replacing $t - \tau + z$ by x ,

$$= L[\int_{t-\tau}^{t} \{F_{k-1}(t-x) - F_k(t-x)\}\, dF(x) + \sum_{n=1}^{\infty} \int_0^{t-\tau} (\int_{t-\xi-\tau}^{t-\xi} \{F_{k-1}(t-\xi-y)$$

$$- F_k(t-\xi-y) - F_k(t-\xi-y)\}\, dF(y))\, d\, F_n(\xi)]$$

replacing $t - \tau + z - \xi$ by y ,

$$= \hat{f}(s) \{ \int_0^{\tau} e^{-sy} F_{k-1}(y)dy - \int_0^{\tau} e^{-sy} F_k(y)dy\}$$

$$+ G(\tau, x) \sum_{n=1}^{\infty} (\hat{f}(s))^n \ ,$$

using Theorem II.B.2 and Corollary II.B.1,

where

$$\begin{cases} \hat{f}(s) = \int_0^{\infty} e^{-st}\, d\, F(t) \ , \\ G(\tau,s) = \int_0^{\tau} e^{-sy} F_{k+1}(y)dy - \int_0^{\tau} e^{-sy} F_k(y)dy + \hat{f}(s) \int_0^{\tau} e^{-sy} F_{k-1}(y)dy \\ \qquad - \hat{f}(s) \int_0^{\tau} e^{-sy} F_k(y)dy \end{cases}$$

$$= \hat{f}(s) \left\{ \int_0^{\tau} e^{-sy} F_{k-1}(y)dy - \int_0^{\tau} e^{-sy} F_k(y)dy \right\} + G(\tau,s) \frac{\hat{f}(s)}{1 - \hat{f}(s)} .$$

Since $\lim_{t \to \infty} P\{D_{(t-\tau,t]} = k\} = \lim_{s \to 0^+} s \cdot L\,[P\{D_{(t-\tau,k]} = k\}]$, the same result achieved in Theorem II.D.3 can be obtained.

In consequence, we can use the result of Theorem II.D.3 to prove the well-known Blackwell's Renewal Theorem, which will be counted as another important example of the Laplace Transform Convolution Theorem applications.

<u>Theorem II.D.4</u>: If the inter-arrival time X is not a lattice random variable and has finite mean μ , then

$$\lim_{t \to \infty} E\{D_{(t-\tau,t]}\} = \lim_{t \to \infty} E\{N_t - N_{t-\tau}\}$$

$$= \lim_{t \to \infty} [m(t) - m(t-\tau)]$$

$$= \frac{\tau}{\mu} , \qquad \text{for every } \tau > 0 .$$

Proof:

Using the result of Theorem II.D.3,

$$\lim_{t \to \infty} E\{D_{(t-\tau,t]}\} = \lim_{t \to \infty} \sum_{k=0}^{\infty} k \cdot P\{D_{(t-\tau,t]} = k\}$$

$$= \sum_{k=1}^{\infty} k \cdot [\lim_{t \to \infty} P\{D_{(t-\tau,t]} = k\}$$

$$= \frac{1}{\mu} \left[\begin{array}{l} \{\int_0^\tau F_0(y)dy - 2\int_0^\tau F_1(y)dy + \int_0^\tau F_2(y)dy\} + 2\{\int_0^\tau F_1(y)dy \\ - 2\int_0^\tau F_2(y)dy + \int_0^\tau F_3(y)dy\} + 3\{\int_0^\tau F_2(y)dy - 2\int_0^\tau F_3(y)dy + \int_0^\tau F_4(y)dy\} \\ + 4\{\int_0^\tau F_3(y)dy - 2\int_0^\tau F_4(y)dy + \int_0^\tau F_5(y)dy\} + \cdots \end{array} \right]$$

$$= \frac{1}{\mu} \int_0^\tau F_0(y)dy = \frac{\tau}{\mu}, \quad \text{since} \quad F_0(y) \equiv 1 \ \text{for} \ y \geq 0 .$$

Finally, Theorems II.C.4, II.D.1 and II.D.3 are put together to give the limiting distribution of the joint distribution of $IP_{t-\tau}$ and $D_{(t-\tau,t]}$.

<u>Theorem II.D.5</u>: Under the assumptions made in Theorem II.C.1,

$$H(j, k) = \lim_{t \to \infty} P\{IP_{t-\tau} = r + j, D_{(t-\tau,t]} = k\}$$

$$= \frac{1}{Q} \cdot \frac{\int_0^\tau F_{k-1}(y)dy - 2\int_0^\tau F_k(y)dy + \int_0^\tau F_{k+1}(y)dy}{\mu},$$

$(i,j = 1, 2, \ldots, Q)$,
for $k = 1, 2, \ldots$

$$= \frac{1}{Q} \cdot \{1 - \frac{1}{\mu}\int_0^{T}[1 - F(x)]dx\} , \quad \text{for} \quad k = 0 .$$

E. Cost Function Formulation for the <Q, r> Model with Backorders and Constant Resupply Lead Time

First of all, we need to discuss the nature of cost factors associated with an inventory system operation to formulate an objective cost function.

The costs incurred in operating an inventory system play a major role in determining what the operating policy (model, or doctrine) should be. There are two types of costs, constant costs and variable costs, which influence the operating policy. Constant costs which are independent of operating doctrines (for example, clerical cost of processing orders) need not be included in any system analysis to determine an optimal operating model. Therefore, only those costs which vary with operating models are necessary for purposes of computing optimal operating models. For example, transportation costs, and receiving and inspection costs are in this category.

Hadley and Whitin (1963) have considered the following five types of relevant costs in determining what the operating policy should be; the costs of procuring the units stocked, the costs of carrying the items in inventory, the costs of filling customers' orders (demands), the costs associated with demands occurring when the system is out of stock, and the costs of operating the data (information) processing system.

The important thing to note about the costs is that they need not be the same as what would be computed from accounting records, because of its varying with the operating doctrine and the components of stock-out costs and carrying costs are not out of pocket costs, but instead represent goodwill costs or opportunity costs.

Some of the costs of filling customers' orders is not depending on the operating doctrine, but varying with the demand rate. These are the costs of the accounting operations, the salaries of those in the warehouse, the costs of packing, and the shipping costs, etc., which need not be considered in the cost study. Rather, the costs arising from the special action required in the case of a customer's demand when the system is out of stock will depend on the operating doctrine, since the fraction of the out-of-stock time in the system will depend on the operating doctrine.

Therefore, the procurement, carrying, and stockout costs, and the cost of operating the information processing system will be considered in this study.

In consideration of the time period over which the system cost is to be computed, the long-run expected average annual cost $\mathcal{C}$ will be formulated to serve as the objective function and its minimization over the long period of time will be the criterion to determine the operating doctrine, since it may be more convenient rather than minimizing the present worth of all future costs. Given that $c(t)$ be the total cost incurred for a time period of length t years, $\mathcal{C}$ is defined as follows:

$$\mathcal{L} = \lim_{t \to \infty} \frac{c(t)}{t} .$$

In the real world, demands can almost never be predicted with certainty; instead they had better be described in probabilistic terms. Realistic inventory models must account for such uncertainty in demand. For example, the mean rate of demand may change with time. Furthermore, the demand rate change may appear in a cyclic fashion.

For this study, the expected values of all relevant random variables will be accounted for to form the function $\mathcal{L}$. Now, we start making assumptions on relevant costs.

The procurement cost is composed of a fixed ordering costs \$A , which is approximately proportional to the number of orders placed, and of a variable cost \$C per unit associated with transportation costs, part of the receiving costs, and part of the inspection costs. Moreover, the unit cost \$C will be assumed independent of the quantity ordered.

For the inventory carrying (holding) costs, the instantaneous rate at which inventory carrying costs are incurred is proportional to the investment in inventory at that point in time. The constant of the proportionality or just the carrying charge, denoted by "I", will be used to estimate the carrying costs. "I" has the dimension of "cost per unit time per monetary unit invested in inventory" (for example, dollars per year per dollar of inventory investment). Therefore, the instantaneous rate of incurring the carrying charges in the units of dollars per year is $IC \cdot x$, where C is the unit cost of each item in dollars and x is the on-hand inventory level. As a matter of fact, the inventory carrying charge "I" is the sum of the carrying charges arising from opportunity costs, pilferage and breakage, insurance costs, taxes, etc. The opportunity cost is not a direct out-of-pocket cost, but incurred by having capital tied up in inventory rather than having it invested elsewhere. Therefore, the opportunity cost is equal to the largest rate of return which the system could obtain from alternative investments.

For the stockout costs, there are two cases such as backorder costs and lost-sales costs incurred by having demands occur when the system is out of stock. The backorder costs are composed of the cost of attempting to find out when the customer's order can be filled and giving him this information, the cost of keeping the system idle for lack of parts, and the factor of customers' goodwill loss. When units are demanded one at a time, a backorder cost will in general be composed of a fixed cost per unit backordered and a varying cost in proportion to the length of time for which the unit remains backordered. Therefore, the cost of each unit backordered can be estimated by $B(t) = B + \hat{B} \cdot t$ a function of the time t for which the backorder remained on the books, where B denotes the fixed cost per unit backordered and $\hat{B}$ represents the varying cost in proportion to the length of time. Denoting "units times years" by "unit years," $\hat{B}$ has the dimension of dollars per unit year of shortage in the case of which we want the cost for a year to come out in dollars.

For the lost sale costs, demands are lost if they occur when the system is out of stock, and hence there is nothing which corresponds to the length of time for which a unit remains backordered. However, the somewhat intangible factors such as goodwill loss have to be accounted for in addition to the profit lost on the unit in not making the sale. The lost sale costs won't be considered in this work.

The costs of operating the information processing system may include such things as the cost associated with having a computer continuously update the inventory records, or the cost of making an actual inventory count, or the cost of making demand predictions. Under the deterministic models for which the rate of demand for units stocked by the system is assumed to be known with certainty and be constant over time, it is possible to determine for all future times precisely what the state of the system will be if the state is known at a given time and if the quantity to be ordered and the reorder point are specified. However, under the assumption of random demand, it is no longer possible to make such predictions with certainty, since the times of occurrence of the demands and also the number of units demanded per demand are random. Therefore, a so-called transactions-reporting system is sometimes equipped to record and report each transaction (demand, placement of order, receipt of shipment, etc.) as it occurs. It is known that the $\langle Q, r \rangle$ model can be optimal if the transactions-reporting system is used and units are demanded one at a time. By the way, this processing system may cost inventory systems too much. Thus, an alternative has been suggested which has the state of the inventory system examined only at discrete, usually equally spaced points in time.

Recall the assumption made in Chapter I that demands occurring when the system is out of stock are backordered, units are demanded one at a time, and procurement lead time is constant, τ .

The inventory position $\{IP_t ; t \geq 0\}$ and the net inventory $\{NIS_t ; t \geq 0\}$ were defined early to be, respectively, the amount on hand $\{OH_t ; t \geq 0\}$ plus on order minus backorders $\{BO_t ; t \geq 0\}$ and the amount on hand minus backorders.

Recall also that under the $<Q, r>$ model a quantity Q is ordered each time the appropriate inventory level (the on-hand inventory, the net inventory, the on-hand plus on-order inventory, or the inventory position) reaches the reorder point r . Therefore, the final objective is to determine the optimal values of Q and r which minimizes the corresponding objective cost function $\mathcal{L}(Q, r)$.

It is important to note that the on-hand inventory or net inventory can not be used to rigorously define r , since a possible heavy demand during some cycle and a huge number of backorders might cause the on-hand inventory never to get back up to r again, and hence another order would never be placed. The inventory position is generally used as a suitable level for defining the reorder point without getting involved with the above difficulties.

When we define the reorder point r in terms of the inventory position, the inventory position becomes $r+Q$ immediately after an order is placed. Thus, the inventory position must have one of the values $r+1, \ldots, r+Q$. It is never in a state r , because as soon as a demand occurs which reduces the inventory position to the state r

an order is placed bringing the state to $r+Q$. By the way, the specification of the inventory position does not directly tell us anything about the on-hand inventory or the net inventory. Therefore, we need to specify the on-hand inventory and the net inventory by use of the inventory position. Let $\{N_t;\ t \geq 0\}$ denote the cumulative counting of demand occurrences by time t. Then $\{N_t\}$ is a discrete-valued continuous-parameter stochastic process (or a renewal process) with simple paths increasing in unit steps. Note that everything on order at time $t-\tau$ will have arrived in the system by time t and nothing not on order at time $t-\tau$ can have arrived in the system by time t. By definition, the next relations follow:

$$\left.\begin{aligned} NIS_t &= IP_{t-\tau} - D_{(t-\tau,t]} \quad \text{for } t \geq \tau \geq 0, \text{ where } D_{(t-\tau,t]} \equiv N_t - N_{t-\tau} \\ &= OH_t - BO_t \\ &\text{and hence} \\ NIS_t &= OH_t, \quad \text{if } NIS_t \geq 0 \\ &= BO_t\ , \text{ otherwise} \end{aligned}\right\} \tag{2.5.1}$$

From the relation of Eq. (2.5.1), if the joint distribution of $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}$ for $t \geq \tau \geq 0$ is determined, then the distribution of $\{NIS_t\}$ can be immediately computed.

With the result of the joint long-run limit distribution of $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ in Theorem II.D.5, we are about to find the long-run limit distribution of $\{NIS_t\}$ which can be used to determine the probability P_{os} that the system is out of stock, the long-run expected on-hand inventory $E[OH]_Q$ and the long-run expected backorders $E[BO]_Q$. This effort will then lead to the formulation of a long-run expected average annual cost function under the assumptions made early on the cost factors, where the minimization of the function is the criterion to determine the optimum Q and r .

It was proved in Theorem II.C.4 that $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ for $t \geq \tau \geq 0$ are mutually independent of each other. We want to introduce the next expression for some later usages; for j = 1, 2, ..., Q ,

$$P\{IP_{t-\tau} = r+j,\ D_{(t-\tau,t]} = j-s\}^{+} = P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = j-s\}^{+}$$

$$= P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = j-s\}, \quad \text{if} \quad j \geq s \qquad (2.5.2)$$

$$= 0 \qquad\qquad , \quad \text{otherwise} .$$

Referring to Eq. (2.5.1),

$$P\{NIS_t = r+s\} = \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j,\ D_{(t-\tau,t]} = j-s\}^{+} ,$$

$$\text{for} \quad S = Q,\ Q-1,\ Q-2,\ \ldots,\ 0,\ -1,\ -2,\ \ldots$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = j-s\}^{+} . \quad (2.5.3)$$

From Eq. (2.5.1) and Eq. (2.5.3),

$$P\{OH_t = x\} = P\{NIS_t = x\}\ , \quad \text{for} \quad x = 0,\ 1,\ 2,\ \ldots$$

$$= \sum_{j=1}^{\Sigma} P\{IP_{t-\tau} = r+j,\ D_{(t-\tau,t]} = r+j-x\}^{+}$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = r+j-x\}^{+}. \qquad (2.5.4)$$

Therefore, the expected on-hand inventory at time t is

$$E[OH]_t = \sum_{x=0}^{\infty} x \cdot P\{OH_t = x\}$$

$$= \sum_{x=0}^{r+Q} x \cdot P\{OH_t = x\}$$

$$= \sum_{x=0}^{r+Q} x \cdot \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = r+j-x\}^{+}$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\} \sum_{x=0}^{r+Q} x \cdot P\{D_{(t-\tau,t]} = r+j-x\}^{+}$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\} \sum_{x=0}^{r+j} x \cdot P\{D_{(t-\tau,t]} = r+j-x\}$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\} \sum_{n=0}^{r+j} (r+j-n)\ P\{D_{(t-\tau,t]} = n\}\ ,$$

where $n \equiv r + j - x$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\} \left[(r+j) \sum_{n=0}^{r+j} P\{D_{(t-\tau,t]} = n\} - \sum_{n=0}^{r+j} n \cdot P\{D_{(t-\tau,t]} = n\}\right], \tag{2.5.5}$$

where $D_{(t-\tau,t]}$ is an arbitrarily distributed random variable and its asymptotic limit distribution is shown in Theorem II.D.3.

The long-run expected number of unit years of on-hand inventory (storage) is

$$\lim_{t \to \infty} E[OH_t] = \lim_{t \to \infty} \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}$$

$$\cdot \left[(r+j) \sum_{n=0}^{r+j} P\{D_{(t-\tau,t]} = n\} - \sum_{n=0}^{r+j} n \cdot P\{D_{(t-\tau,t]} = n\}\right]$$

$$= \sum_{j=1}^{Q} \left[\lim_{t \to \infty} P\{IP_{t-\tau} = r+j\}\right]$$

$$\cdot \lim_{t \to \infty} \left[(r+j) \sum_{n=0}^{r+j} P\{D_{(t-\tau,t]} = n\} - \sum_{n=0}^{r+j} n \cdot P\{D_{(t-\tau,t]} = n\}\right]$$

$$= \sum_{j=1}^{Q} \cdot \frac{1}{Q} \left[(r+j) \sum_{n=0}^{r+j} \left(\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\}\right)\right.$$

$$\left. - \sum_{n=0}^{r+j} n \cdot \left(\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\}\right)\right], \tag{2.5.6}$$

by Theorem II.D.1, where $\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\}$ can be evaluated by use of Theorem II.D.3.

Thus, the long-run expected average number of unit years of on-hand inventory incurred per year, denoted by $E[OH]_Q$, follows:

$$E[OH]_Q = \lim_{t \to \infty} \frac{\int_0^{\tau} E\,[OH_t]\,dt}{t}$$

$$\doteq \lim_{t \to \infty} E\,[OH_t]\,. \qquad (2.5.7)$$

Likewise, from Eq. (2.5.1) and Eq. (2.5.3),

$$P\{BO_t = x\} = P\{NIS_t = -x\}\ , \quad \text{for} \quad x = 1,\ 2,\ \ldots,$$

$$= \sum_{j=1}^{\Sigma} P\{IP_{t-\tau} = r+j,\ D_{(t-\tau,t]} = r+j+x\}$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = r+j+x\}. \quad (2.5.8)$$

Therefore, the expected number of backorders on the books at any time t or the expected number of unit years of shortage at any time t is

$$E[BO_t] = \sum_{x=1}^{\infty} x \cdot P\{BO_t = x\}$$

$$= \sum_{x=1}^{\infty} x \cdot \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\ P\{D_{(t-\tau,t]} = r+j+x\}$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\} \sum_{x=1}^{\infty} x \cdot P\{D_{(t-\tau,t]} = r+j+x\}$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\} \sum_{n=r+j+1}^{\infty} (n-r-j)\ P\{D_{(t-\tau,t]} = n\}\ ,$$

where $n \equiv r + j + x$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\Big[\sum_{n=r+j+1}^{\infty} n \cdot P\{D_{(t-\tau,t]} = n\}$$

$$- \sum_{n=r+j+1}^{\infty} (r+j)\ P\{D_{(t-\tau,t]} = n\}\Big]$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\Big[E[D_{(t-\tau,t]}] - \sum_{n=0}^{r+j} n \cdot P\{D_{(t-\tau,t]} = n\}$$

$$- \sum_{n=r+j+1}^{\infty} (r+j)\ P\{D_{(t-\tau,t]} = n\}\Big]$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\Big[E[D_{(t-\tau,t]}] - \sum_{n=0}^{r+j} n \cdot P\{D_{(t-\tau,t]} = n\}$$

$$- (r+j)\Big(1 - \sum_{n=0}^{r+j} P\{D_{(t-\tau,t]} = n\}\Big)\Big]$$

$$= \sum_{j=1}^{Q} P\{IP_{t-\tau} = r+j\}\Big[E[D_{(t-\tau,t]}] - (r+j) + \sum_{n=0}^{r+j} (1-n) P\{D_{(t-\tau,t]} = n\}\Big]. \tag{2.5.9}$$

The long-run expected number of unit years of backorders (shortage) is

$$\lim_{t \to \infty} E[BO_t] = \lim_{t \to \infty} \sum_{j=1}^{Q} P\{IP_{t-\tau} = r + j\} [E[D_{(t-\tau,t]}]$$

$$- (r+j) + \sum_{n=0}^{r+j} (1-n) P\{D_{(t-\tau,t]} = n\}]$$

$$= \sum_{j=1}^{Q} [\lim_{t \to \infty} P\{IP_{t-\tau} = r + j\} \lim_{t \to \infty} [E[D_{(t-\tau,t]}]$$

$$- (r+j) + \sum_{n=0}^{r+j} (1-n) P\{D_{(t-\tau,t]} = n\}]$$

$$= \sum_{j=1}^{Q} [\lim_{t \to \infty} P\{IP_{t-\tau} = r + j\}] \cdot [\lim_{t \to \infty} E[D_{(t-\tau,t]}]$$

$$- (r+j) + \sum_{n=0}^{r+j} (1-n) (\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\})]$$

$$= \sum_{j=1}^{Q} \cdot \frac{1}{Q} [\frac{\tau}{\mu} - (r+j) + \sum_{n=0}^{r+j} (1-n)(\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\})], \tag{2.5.10}$$

by Theorem II.D.1 and Theorem II.D.4, and also

$\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\}$ can be evaluated by Theorem II.D.3.

Thus, the long-run expected average number of unit years of backorders incurred per year, denoted by $E[OB]_Q$, is

$$E[BO]_Q = \lim_{t \to \infty} \frac{\int_0^{\tau} E[BO_t]dt}{t}$$

$$\doteq \lim_{t \to \infty} E[BO_t] \; . \qquad (2.5.11)$$

The long-run expected average value of the random variable, say ΔBO_t , representing the number of backorders incurred between time $t - \tau$ and t can be approximately computed by multiplying the mean rate of demand (or demand intensity) λ by the out-of-stock probability P_{os} . Denote by $P_{os}(t)$ the probability that the system is out of stock at time t . Then, the limit out-of-stock probability P_{os} follows:

$$P_{os} = \lim_{t \to \infty} P_{os}(t)$$

$$= \lim_{t \to \infty} \sum_{x=1}^{\infty} P\{BO_t = x\}$$

$$= \lim_{t \to \infty} \sum_{x=1}^{\infty} \sum_{j=1}^{Q} P\{IP_{t-\tau} = r + j\} \, P\{D_{(t-\tau,t]} = r + j + x\} \; ,$$

from Eq. (2.5.8)

$$= \lim_{t \to \infty} \sum_{j=1}^{Q} P\{IP_{t-\tau} = r + j\} \sum_{n=r+j+1}^{\infty} P\{D_{(t-\tau,t]} = n\} \; ,$$

where $n = r + j + x$

$$= \lim_{t \to \infty} \sum_{j=1}^{Q} P\{IP_{t-\tau} = r + j\} [1 - \sum_{n=0}^{r+j} P\{D_{(t-\tau,t]} = n\}]$$

$$= \sum_{j=1}^{Q} [\lim_{t \to \infty} P\{IP_{t-\tau} = r + j\}] [1 - \sum_{n=0}^{r+j} (\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\})]$$

$$= \sum_{j=1}^{Q} \frac{1}{Q} [1 - \sum_{n=0}^{r+j} (\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\})] , \qquad (2.5.12)$$

by Theorem II.D.1.

The mean rate of demand λ is formally defined as follows:

$$\lambda = \lim_{\Delta t \to 0} \frac{E[D_{(t,t+\Delta t]}]}{\Delta t} . \qquad (2.5.13)$$

Hence, the long-run expected average number of backorders incurred per year, denoted by $E[\Delta BO]_Q$, is

$$B[\Delta BO]_Q \doteq \lim_{t \to \infty} E[\Delta BO_t]$$

$$= \lim_{t \to \infty} \lambda \cdot P_{os}(t)$$

$$= \lambda \cdot P_{os}$$

$$= \lambda \cdot \frac{1}{Q} \sum_{j=1}^{Q} [1 - \sum_{n=0}^{r+j} (\lim_{t \to \infty} P\{D_{(t-\tau,t]} = n\})], \qquad (2.5.14)$$

by Eq. (2.5.12).

Moreover, since the mean rate of demand is λ units per year and each order quantity is Q , the number of orders placed per year must average to λ/Q .

All the terms meeded for the long-run expected average annual cost expression $\mathcal{L}(Q, r)$ have been evaluated. With the cost parameters discussed early in this section, it is formulated as follows;

$$\mathcal{L}(Q, r) = \frac{\lambda}{Q} \cdot A + IC \cdot E[OH]_Q + B \cdot E[\Delta BO]_Q + \hat{B} \cdot E[BO]_Q \quad . \qquad (2.5.15)$$

III. PERIODIC REVIEW

A. Introduction

The intention of this chapter is to combine the recent work on the nonstationary Markov chains with the classical models of inventory theory to derive the cost functions of an inventory system under certain operating doctrines in the face of nonstationary Poisson demand.

It is not always desirable to have inventory systems use transactions reporting review procedure, since it may be too costly. The periodic review procedure is an alternative. When the procedure is used, the state of the inventory system is examined only at discrete points in time, since decisions such as whether or not to place an order are made only at the review times and thus the decision makers do not know anything about the state of the system at times other than the review times.

Three operating doctrines, "Rr" doctrine, "order up to R" doctrine and "nQ" doctrine, are commonly used for the periodic review inventory systems. These terminologies are adapted from Hadley and Whitin (1963). Symbolically, those doctrines are referred to, respectively, as <R, r, T>, <R, T> and <nQ, r, T> models. Under the <R, T> model, an order is placed at each review time if any units have been demanded in the past period, so that the ordered quantity can vary from one review period to the next. According to the <R, r, T> model, a procurement of sufficient quantity which bring the inventory levels up to R is made at a review time only if the inventory position in the backorders case is less than or equal to r. An integral multiple of some fundamental quantity Q (i.e., nQ for $n = 1, 2, \ldots$), in $<nQ, r, T>$ model, is ordered at a review time only if the inventory position or the amount on hand plus on order at the review time is less than or equal to r. It is stated in Hadley and Whitin (1963) that a $<R, r, T>$ model is usually the optimal one, if all demands occurring when a system is out of stock are backordered. We know that $<nQ, r, T>$ and $<R, T>$ models are only approximations to the optimal $<R, r, T>$ model, and further, that the $<R, T>$ model is a special case of the $<nQ, r, T>$ model and also of the $<R, r, T>$ model. Therefore, once having obtained the precise equations for the $<nQ, r, T>$ model, we can immediately get the exact equations for a $<R, T>$ model under the same assumptions which apply in deriving the $<nQ, r, T>$ model. Even if the $<R, T>$ model is widely used in practice for periodic review systems, the $<nQ, r, T>$ and $<R, r, T>$ models will be dealt with under the assumptions made in Chapter I for this study.

Before investigating the Markov property of an inventory process $\{IP_{T_k} ; T_k \geq 0 \text{ for } k = 0, 1, 2, \ldots\}$, where $T_0 \equiv 0$, we want to define some important terminologies. A stochastic process $\{N_t, 0 \leq t < \infty\}$ representing the number of demand occurrences by time t is said to have "independent increment" if the random variables $D_{(t_{n-1}, t_n]}$ are independent, where $D_{(t_{n-1}, t_n]} \equiv N_{t_n} - N_{t_{n-1}}$ for all choices of indices $t_0 < t_1 < \ldots < t_n$. In addition, if $D_{(t_{n-1}+h, t_n+h]}$ has the same distribution as $D_{(t_{n-1}, t_n]}$ for $h > 0$ and $n = 1, 2, \ldots$, it is said that the process $\{N_t\}$ has "stationary independent increments." Otherwise, the process is said to have "nonstationary independent increments."

As was pointed out earlier, in the inventory system under study the inventory position IP_{T_k} at a review time $T_k > 0$ $(k = 1, 2, \ldots)$ can be determined only by the inventory position immediately after the preceding review time T_{k-1}, and the accumulated demand during the k^{th} review period $(T_{k-1}, T_k]$. Under the assumption of independent demand events in each different period, it is reasonable to assume that the process $\{IP_{T_k} ; T_k \geq 0 \text{ for } k = 0, 1, 2, \ldots\}$ satisfies the Markov property that the future system development is completely determined by the present state and is independent of the whole past history.

Under the assumption of a Poisson demand process, the corresponding stationary Markov chain $\{IP_{T_k} ; k = 0, 1, 2, \ldots\}$ with a constant review interval T, where $T \equiv T_{k+1} - T_k$, and finite state spaces $S = \{r+1, r+2, \ldots, r+Q\}$ for the $<nQ, r, T>$ model and $S = \{r+1, r+2, \ldots, R\}$ for the $<R, r, T>$ model, has been studied by Hadley and Whitin (1963). The stationary Markov chain means that the conditional probability of the inventory position being $r+j$ at the next review time, given the inventory position $r+i$ at one review time, does not depend on time parameter T_k, that is, for $n > 0$

$$P\{IP_{kT} = r+j \mid IP_{(k-1)T} = r+i\} = P\{IP_{(k+n)T} = r+j \mid IP_{(k-1+n)T} = r+i\}$$

$$\equiv p_{ij} \quad \text{(say)} .$$

They did construct the corresponding stationary finite transition matrix of $\{P_{ij}\}$, $(i,j \in S)$, for the $\langle R, r, T\rangle$ model, but in view of computational difficulty, the matrix was not directly used to find the long-run distribution (or the invariant probability) of the Markov chain $\{IP_{kT}\}$. It has been pointed out by Mettanant (1977) that it is not hard to solve for the long-run distribution. However, they gave the very complicated closed form of solutions for the long-run limit distributions of $\{IP_{kT}\}$. Therefore, the issue is taken up below in the general and simpler setting of the long-run limit distribution from finite transition matrices under the $\langle R, r, T\rangle$ model.

In this chapter, we shall investigate the implications for the nonstationary inventory position process $\{IP_{T_k} ; T_k \geq 0 \text{ for } k = 0, 1, 2, \ldots\}$, and hence the full inventory system, of a demand process which is a general nonhomogeneous Poisson process, with intensity function $\lambda(t)$ replacing the usual constant intensity λ, whence the concept of the weak and strong ergodicities of nonstationary Markov chains will be applied to determine the relevant limit distribution of the inventory position $\{IP_{T_k}\}$ corresponding to the $\langle nQ, r, T\rangle$ and $\langle R, r, T\rangle$ models. Once such a limit distribution is determined, the annual inventory system operation cost evaluation seems straightforward. The significance of this approach is that this analysis treats a more realistic stochastic demand process. It should be noted in addition that it will be assumed that the lead time τ is constant.

The mean value function $m(t) \equiv E\{N_t\}$ of a nonstationary Poisson process is always assumed to be continuous and also usually differentiable, with derivative $\lambda(t) = d\, m(t)/dt$, where $\lambda(t)$ is called the intensity function. It is a useful fact that the Poisson differential assumptions, with mean rate (or intensity) λ replaced by $\lambda(t)$, yield Poisson demand in time interval $(0, t]$. It will further be shown below that the intensity function $\lambda(t)$ of the nonstationary Poisson process yields the parameters of the nonstationary Markov chain process for inventory positions, under the $<nQ, r, T>$ and $<R, r, T>$ models, leading to the relevant limit distributions.

This chapter will also include an approach to both the $<nQ, r, T>$ and the $<R, r, T<$ models which takes into account possible cyclic behavior (for example, seasonal trend) of demand. This case is modeled by a cyclic nonstationary Markov chain, for which Cesaro ergodicity pertains. Bowerman, David and Isaacson (1977) have verified sufficient conditions for the strong ergodicity of a nonstationary Markov chain in which the transition matrices repeat themselves in a cyclic fashion. This weaker form of ergodicity is still sufficient for the computation of long-run expected average cost.

This section will be followed by Section B for a discussion of Markov Chain Theory, Section C for the long-run limit distribution computation of $\{IP_{T_k}\}$ and Section D for the derivation of long-run expected average annual inventory system operation cost function. Thereafter, only the thing to do is to determine the optimal values of Q, r, T and R which minimizes the relevant cost function corresponding to each operating model. This job can normally be done on a digital computer.

B. Long-Run Behavior of Finite Nonstationary Markov Chains

Some discrete-parameter stochastic processes $\{X_t; t = 0, 1, 2, \ldots\}$ have the outcome functions $\{X_t(\omega)\}$ with $\omega \in \Omega$ (sample space) which range over the elements of a countable state space $S = \{1, 2, 3, \ldots\}$. A finite discrete-parameter stochastic process is a stochastic process $\{X_t; t = 0, 1, 2, \ldots\}$ for which all the outcome functions $\{X_t(\omega); \omega \in \Omega\}$ range over the elements of a finite state space $S = \{1, 2, \ldots, N\}$. There are some discrete-parameter stochastic processes satisfying the Markov Chain property, which is the basis for work in this study. We shall begin this section with given the formal definition of Markov Chain.

Definition III.B.1. A stochastic process $\{X_t; t = 0, 1, 2, \ldots\}$ with a finite or countable state space $S = \{1, 2, \ldots.\}$ is said to be a Markov chain if for all states $i_0, i_1, \ldots, i_t$ and for $t \geq 1$;

$$P\{X_t = i_t \mid X_0 = i_0, X_1 = i_1, \ldots, X_{t-1} = i_{t-1}\} = P\{X_t = i_t \mid X_{t-1} - i_{t-1}\}. \tag{3.2.1}$$

A finite Markov chain is a stochastic process $\{X_t; t = 0, 1, 2, \ldots\}$ with a finite state space $S = \{1, 2, \ldots, N\}$ satisfying the relation of Eq. (3.2.1). Eq. (3.2.1) means that the transition of a Markov chain from time $t-1$ to t is determined only by the conditional probability $P\{X_t = i_t \mid X_{t-1} = i_{t-1}\}$.

If we denote by $p_{ij}^{t-1,t} = P\{X_t = j \mid X_{t-1} = i\}$ the one-step transition probability from state i to state j on the t^{th} step, the one-step transition matrix of a Markov chain with state space $S = \{1, 2, \ldots\}$ is defined for $t \geq 1$ to be;

$$P^{t-1,t} = \begin{bmatrix} p_{11}^{t-1,t} & p_{12}^{t-1,t} & \cdots\cdots & p_{1j}^{t-1,t} & \cdots\cdots \\ p_{21}^{t-1,t} & p_{22}^{t-1,t} & \cdots\cdots & p_{2j}^{t-1,t} & \cdots\cdots \\ \vdots & \vdots & \cdots\cdots & \vdots & \cdots\cdots \\ p_{i1}^{t-1,t} & p_{i2}^{t-1,t} & \cdots\cdots & p_{ij}^{t-1,t} & \cdots\cdots \\ \vdots & \vdots & \cdots\cdots & \vdots & \cdots\cdots \end{bmatrix}$$

where $p_{ij}^{t-1,t} \geq 0 \quad \forall \; i \in S, \; \forall \; j \in S,$ and

$\sum_{j \in S} p_{ij}^{t-1,t} = 1, \quad \forall \; i \in S$.

Definition III.B.2. If $p_{ij}^{t-1,t}$ is independent of t , then the Markov chain is said to possess stationary transition probabilities and is called a stationary (or homogeneous) Markov chain. If $p_{ij}^{t-1,t}$ is dependent upon t , then the Markov chain is called a nonstationary (or nonhomogeneous) Markov chain.

Since the transition matrix $P^{t-1,t}$ of a stationary Markov chain has components $\{p_{ij}^{t-1,t}\}$ satisfying

$$p_{ij}^{t-1,t} = P\{X_t = j \mid X_{t-1} = i\}$$

$$= P\{X_{t+u} = j \mid X_{t-1+u} = i\} \quad \text{for } u \geq 0 \text{ , } \forall \; i \in S \text{ and } \forall \; j \in S \text{ ,}$$

we write $P^{t-1,t}$ as P for $t \geq 1$.

Example III.B.1: Let $\{X_t ; t = 0, 1, 2, \ldots\}$ be a Markov chain having probability transition matrix from time t-1 to time t of; for $t \geq 1$,

$$P^{t-1,t} = \begin{bmatrix} 0.8 - 0.1/t & 0.1 + 0.2/t & 0.1 - 0.1/t \\ 0.6 - 0.1/t & 0.3 - 0.1/t & 0.1 + 0.2/t \\ 0.7 - 0.1/t & 0 & 0.3 - 0.1/t \end{bmatrix} .$$

Then $\{X_t ; t = 0, 1, 2, \ldots\}$ is a nonstationary Markov chain.

<u>Theorem III.B.1</u>. (Chapman-Kolmogorov Identity): For all nonnegative integers m and n and state space $S = \{1, 2, \ldots\}$

$$p_{ij}^{m+n} = \sum_{k \in S} p_{ij}^{m} \cdot p_{kj}^{n} \text{ .}$$

It we denote by $P^{(n)}$ the matrix of n-step transition probabilities $p_{ij}^{n} = P\{X_n = j \mid X_0 = i\}$, then Chapman-Kolmogorov Identity asserts that $P^{(m+\ell)} = P^{(m)} \cdot P^{(\ell)}$, which is reduced to $P^{(n)} = P^{(m)} \cdot P^{(n-m)}$

with the replacement of $(m+\ell)$ by n for any $m \leq n$. Hence, $P^{(n)} = P \cdot P^{(n-1)} = P \cdot (P \cdot P^{(n-2)}) = \ldots = P^n$. In general, $p_{ij}^{(m,m+n)} = P\{X_{m+n} = j \mid X_m = i\}$ is the $(i,j)^{th}$ entry of $P^{(m,m+n)} = P^{m,m+1} \cdot P^{m+1,m+2} \ldots P^{m+n-1,m+n}$ for $n \geq 1$. This leads to the following theorem.

<u>Theorem III.B.2</u>: For $t > n \geq m \geq 0$.

$$p_{ij}^{(m,t)} = \sum_{k \in S} p_{ik}^{(m,n)} \cdot p_{kj}^{(n,t)} .$$

Let $f^{(0)} = (f_1^{(0)}, f_2^{(0)}, \ldots, f_N^{(0)})$ be a starting vector possessing the property of $\sum_{i=1}^{N} f_i^{(0)} = 1$ and $f_i^{(0)} \geq 0 \quad \forall\, i$, where $f_i^{(0)} = P\{X_0 = i\}$. $f_i^{(0)}$ is the probability distribution that a process $\{X_t;\ t = 0, 1, 2, \ldots\}$ starts at state i . If a sequence of transition matrices $\{P_n\}_{n=1}^{\infty}$, where $P_n = P^{n-1,n}$, and $f^{(0)}$ are given, the probabilities of various outcomes of a finite nonstationary Markov chain $\{X_t;\ t = 0, 1, 2, \ldots\}$ with state space $S = \{1, 2, \ldots, N\}$ can be determined as follows; for $j \in S$

$$\text{i)}\quad P\{X_1 = j\} = \sum_{i=1}^{N} P\{X_0 = i\}\, P\{X_1 = j \mid X_0 = i\}$$

$$= \sum_{i=1}^{N} f_i^{(0)} \cdot p_{ij}^{(0,1)} \quad . \qquad (3.2.2)$$

ii) $$P\{X_2 = j\} = \sum_{k=1}^{N} P\{X_1 = k\}\, P\{X_2 = j \mid X_1 = k\}$$

$$= \sum_{k=1}^{N} \Big(\sum_{i=1}^{N} P\{X_0 = i\}\, P\{X_1 = k \mid X_0 = i\} \Big)$$

$$\cdot\, P\{X_2 = j \mid X_1 = k\}$$

$$= \sum_{k=1}^{N} \sum_{i=1}^{N} f_i^{(0)}\, p_{ik}^{(0,1)}\, p_{kj}^{(1,2)} \,, \qquad (3.2.3)$$

and so on.

Therefore, the distribution of where the process is situated after n steps can be found from $f^{(n)} = f^{(0)} \cdot P_1 \cdot P_2 \cdots P_n$, since

$$P\{X_0 = i_0,\, X_1 = i_1,\, \ldots,\, X_n = i_n\}$$

$$= P\{X_0 = i_0\}\, P\{X_1 = i_1 \mid X_0 = i_0\}\, P\{X_2 = i_2 \mid X_0 = i_0,\, X_1 = i_1\} \cdots$$

$$P\{X_n = i_n \mid X_0 = i_0,\, X_1 = i_1,\, \ldots,\, X_{n-1} = i_{n-1}\}$$

$$= f_{i_0}^{(0)} \prod_{m=1}^{n} p_{i_{m-1} i_m}^{m-1,m} \,.$$

In some nonstationary Markov chains the vector $f^{(n)}$ converges to a fixed vector π which is independent of the starting vector $f^{(0)}$. The limit vector π is called the long-run distribution or the invariant probability distribution.

Let $f^{(m,n)} = f^{(0)} \cdot P_{m+1} \cdot P_{m+2} \cdots P_{n-1} \cdot P_n$. $f^{(m,n)}$ may converge to a fixed vector π for all m so that no matter when the process starts, whether at time zero or time n-1, $\lim_{n \to \infty} f^{(m,n)} = \pi$ independently of $f^{(0)}$. This convergence shows that the effect of $f^{(0)}$ is lost after a long time, so that it is often referred to as loss of memory with convergence. On the other hand, some vector $f^{(n)}$ may not always be possible to have the behavior of both convergence and loss of memory together. For example, the probability of being in a particular state in n steps may be eventually independent of the initial state, but dependent on time n . This kind of behavior is referred to as loss of memory without convergence.

The necessary and sufficient conditions for these two different long-run behaviors of nonstationary Markov chains can be formally established using the ergodic coefficient $\alpha(P_n)$ for $n \geq 1$ which has been defined by Dobrushin (1956). The applications of the ergodic coefficient to stationary Markov chains are rather simple corollaries of results that relate to nonstationary Markov chains.

<u>Definition III.B.3</u>:

a) A matrix A whose $(i,j)^{th}$ element is denoted by a_{ij} is called a stochastic matrix if $a_{ij} \geq 0$, $\forall\, i$ and $\forall\, j$ and $\sum_{j \in S} a_{ij} = 1$, $\forall\, i$.

b) A matrix A is called a doubly stochastic matrix if $a_{ij} \geq 0$, $\forall\, i$ and $\forall\, j$, $\sum_{i \in S} a_{ij} = 1$ $\forall\, j$ and $\sum_{j \in S} a_{ij} = 1$, $\forall\, i$.

c) A matrix A is called a primitive matrix if $a_{ij} \geq 0$ and $a_{ij}^{m} > 0$ for $m \geq 1$, $\forall i$ and $\forall j$.

Therefore, if the transition matrix P of a Markov chain with $S = \{1, 2, \ldots, N\}$ is primitive, then after sufficient lapse of time the chain can stay in any state of the space, no matter which of the states it started in.

The transition probability matrices of inventory position process $\{IP_{T_k} ; T_k \geq 0$ for $k = 0, 1, 2, \ldots\}$ under $<nQ, r, T>$ and $<R, r, T>$ operating models form, respectively, a finite positive doubly stochastic matrix and a nonnegative stochastic matrix with two positive columns and two positive rows. Therefore, finite primitive stochastic matrices will be mainly dealt with throughout this chapter.

Let's define the norm of a vector $f = (f_1, f_2, \ldots)$ and the norm of a square matrix $A = \{a_{ij}\}$ by, respectively, $\|f\| = \sum_{i \in S} |f_i|$ and $\|A\| = \sup_{i \in S} \sum_{j \in S} |a_{ij}|$, where $S = \{1, 2, \ldots\}$. Then, we will give the definition of another coefficient $\delta(P)$, called the δ-coefficient of a stochastic matrix P, which has been used more frequently and conveniently than $\alpha(P)$.

<u>Definition III.B.4</u>: If P is a stochastic matrix whose $(i,j)^{th}$ element is p_{ij}, then the delta coefficient of P is defined by

$$\delta(P) = 1 - \alpha(P)$$

$$= \sup_{i,j} \sum_{k \in S} (p_{ik} - p_{jk})^+$$

$$= \left(\frac{1}{2}\right) \sup_{i,j} \sum_{k \in S} |p_{ik} - p_{jk}| ,$$

where $(p_{ik} - p_{jk})^+ = \max(0, p_{ik} - p_{jk})$ and $0 \leq \delta(P) \leq 1$.

The following theorem which has been proved by Paz (1970) is just stated without proof.

Theorem III.B.3: If A and B are matrices such that $A + B$ and AB are well-defined, then

a) $\|A + B\| \leq \|A\| + \|B\|$

b) $\|A \cdot B\| \leq \|A\| + \|B\|$

c) $\delta(AB) \leq \delta(A)\, \delta(B)$

d) $\|kA\| = |k|\, \|A\|$, where k is a constant.

We now introduce an important lemma which can be used to prove the strong ergodicity of a nonstationary Markov chain. Its proof can be found in Isaacson and Madsen (1976) (see also Paz (1971)).

Lemma III.B.1: If P be a stochastic matrix and R be a matrix of the same dimension as P such that $\sum_{j \in S} r_{ij} = 0$, $\forall j$ and $R \leq \infty$, then

$$\|R \cdot P\| \leq \|R\| \cdot \delta(P) .$$

We will also state without proof the next theorem in Isaacson and Madsen (1976) in which the ergodic coefficient can be used to determine the weak ergodicity of a nonstationary Markov chain and the strongly ergodic behavior will be related to the transition matrices rather than the starting vectors.

Theorem III.B.4: Let $\{X_t;\ t = 0, 1, 2, \ldots\}$ be a Markov chain whose transition matrix from time $t-1$ to time t is $P^{t-1,t}$ for $t \geq 1$.

a) The chain is called weakly ergodic if and only if for all $m \geq 0$ $\cdot$ $\lim_{n \to \infty} \delta(P^{(m,n)}) = 0$.

b) The chain is called strongly ergodic if and only if there exists a stochastic matrix G with constant rows (or a constant stochastic matrix) such that for all $m \geq 0$,

$$\lim_{n \to \infty} \| P^{(m,n)} - G \| = 0 .$$

This theorem indicates that weakly ergodic Markov chains have the long-run behavior of "loss of memory without convergence" and strongly ergodic Markov chains have the "loss of memory with convergence" behavior after a long time.

Mott (1957) has verified a sufficient condition for a nonstationary finite Markov chain to be weakly ergodic with $\sigma^t = \max_{j \in S} \{\sigma_j^t\}$ on the state space $S = \{1, 2, \ldots, N\}$, where $\{\sigma_j^t\}$ denotes a least element of the j^{th} column of a transition probability matrix $P^{t-1,t}$.

Corollary III.B.1: A nonstationary finite Markov chain is weakly ergodic if

$$\prod_{t=1}^{n} (1 - \sigma^t) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

This corollary means that a nonstationary finite Markov chain is weakly ergodic if at least one column of $P^{t-1,t}$ is uniformly bounded above zero, that is, $\sigma^t \geq \sigma \geq 0$ for all t. This condition is sometimes more conveniently used to determine the weak ergodicity for nonstationary finite Markov chains.

Note that for finite stationary Markov chains weak ergodicity is equivalent to strong ergodicity.

We will now introduce a theorem which gives sufficient condition for a nonstationary Markov chain to be strongly ergodic. It relates strongly ergodic behavior to a nonnegative left eigenvector corresponding to the eigenvalue 1 rather than the transition matrices. The proof appears in Madsen and Isaacson (1973).

Theorem III.B.5: Let $\{X_t;\ t = 0, 1, 2, \ldots\}$ be a weakly ergodic Markov chain whose transition matrix from time $t-1$ to time t is $P^{t-1,t}$. If for each $t \geq 1$ there exists a row vector π^t such that $\pi^t P^{t-1,t} = \pi^t$, $\|\pi^t\| = 1$, and $\sum_{t=1}^{\infty} \|\pi^{t-1} - \pi^t\| < \infty$, then the Markov chain is strongly ergodic and the strong long-run distribution of the Markov chain is π where $\lim_{t \to \infty} \| \pi^t - \pi \| = 0$.

Perron-Frobenius' theorems stated in Gantmacher (1959) assert that a nonnegative matrix A is primitive if and only if it has a unique maximal eigenvalue λ_0 in its absolute value, that is, if λ_i, $i = 1, 2, \ldots,$ is some other eigenvalue of matrix A, then $|\lambda_i| < |\lambda_0|$. Moreover, λ_0 is a positive, simple root of the characteristic equation and the corresponding eigenvector is positive. It is followed by a lemma, the proof of which appears in Isaacson and Madsen (1976).

Lemma III.B.2: The value 1 is not only an eigenvalue of all finite stochastic matrix P, but also it is the largest eigenvalue of P.

These lead to the following theorem. The reader is also referred to Kemeny and Snell (1960), and Isaacson and Madsen (1976).

Theorem III.B.6: If a $N \times N$ stationary transition matrix P is primitive, then the powers P^m for $m \geq 1$ approach a constant stochastic matrix G such that each row of G is the unique probability vector $\pi = \{\pi_1, \pi_2, \ldots, \pi_N\}$ satisfying $\pi P = \pi$ and hence $PG = GP = G$.

Proof:

The proof is straightforward from Lemma III.B.2 and Theorem III.B.5.

A doubly stochastic matrix is a special type of general transition matrices. In view of Theorem III.B.6, a unique long-run distribution can be obtained from the transition probability matrices of inventory position process $\{IP_{T_k}; T_k \geq 0 \text{ for } k = 0, 1, 2, \ldots\}$ under the $<nQ, r, T>$ model.

Proposition III.B.1: Let $P^{t-1,t}$ and $P^{t,t+1}$ be $N \times N$ doubly stochastic matrices. Then, $A^t \equiv P^{t-1,t} \cdot P^{t,t+1}$ is also a $N \times N$ doubly stochastic matrix.

Proof:

Let a_{ij}^{j}, $p_{ij}^{t-1,t}$ and $p_{ij}^{t,t+1}$ be the $(i,j)^{th}$ components of matrices, respectively, A^t, $P^{t-1,t}$, and $P^{t,t+1}$. Then $A^t \equiv P^{t-1,t} \cdot P^{t,t+1}$ implies that $\forall$ i and $\forall$ j ,

$$a_{ij}^{t} = \sum_{k=1}^{N} p_{ik}^{t-1,t} \cdot p_{kj}^{t,t+1} ,$$

$$\sum_{j=1}^{N} a_{ij}^{t} = \sum_{j=1}^{N} \sum_{k=1}^{N} p_{ik}^{t-1,t} \cdot p_{kj}^{t,t+1}$$

$$= \sum_{k=1}^{N} p_{ik}^{t-1,t} \left(\sum_{j=1}^{N} p_{kj}^{t,t+1} \right)$$

$$= 1 ,$$

and

$$\sum_{i=1}^{N} a_{ij}^{t} = \sum_{i=1}^{N} \sum_{k=1}^{N} p_{ik}^{t-1,t} \cdot p_{kj}^{t,t+1}$$

$$= \sum_{k=1}^{N} p_{kj}^{t,t+1} \left(\sum_{i=1}^{N} p_{ik}^{t-1,t} \right)$$

$$= 1 .$$

Therefore, the proof is complete.

<u>Corollary III.B.2</u>: If the doubly stochastic matrix P for a stationary finite Markov chain with N states is primitive, then the long-run distribution is the unique uniform probability vector $\pi = \{ \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N} \}$.

Proof:

The proof is straightforward from Theorem III.B.6 since $N_i^t = 1$ for $i = 1, 2, \ldots, N$ and $t \geq 1$.

It has been stated in the book of Isaacson and Madsen (1976) that weak ergodicity of nonstationary Markov chains with doubly stochastic matrices is the necessary and sufficient condition for the strong ergodicity. This is a simple corollary of Theorem III.B.5. However, since it is so important claim for nonstationary Markov chains corresponding to the $\langle nQ, r, T \rangle$ inventory system operating policy, we shall present the following theorem.

<u>Theorem III.B.7</u>: Let $\{X_n\}$ be a finite nonstationary Markov chain with state space $S = \{1, 2, \ldots, N\}$. If the sequence of the corresponding transition probability matrices $\{P_n\}_{n=1}^{\infty}$ are all doubly stochastic, then the chain is strongly ergodic with $\pi = \{ \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N} \}$ if and only if it is weakly ergodic.

Proof:

Define by $P^{(m)} = P_1 \cdot P_2 \cdot P_3 \cdots P_m$ for $m = 1, 2, \ldots$ It follows from Proposition III.B.1 that $P^{(m)}$ is also a $N \times N$ doubly stochastic matrix. Let G be a finite, constant stochastic matrix with identical row $\pi = \{ \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N} \}$. Under the assumption that the chain is weakly ergodic, we can show that $P^{(m)}$ tends to G as $m \longrightarrow \infty$ as follows; for $m \geq 1$,

$$\|P^{(m)} - G\| = \|P_1 \cdot P_2 \cdots P_m - G\|$$

$$= \|(P_1 - G) \cdot (P_2 - G) \cdots (P_m - G)\|$$

$$= \|(P_1 - G) \cdot (P_2 \cdot P_3 \cdots P_m - G)\|$$

$$= \|(P_1 - G) \cdot (P_2 \cdot P_3 \cdots P_m)\|, \quad \text{since} \quad (P_1 - G)\cdot G = 0$$

$$\leq \|P_1 - G\| \cdot \delta(P^{1,m}) \quad \text{from Lemma III.B.1.}$$

Therefore, since the chain is assumed to be weakly ergodic so that $\delta(P^{(1,m)}) \longrightarrow 0$ as $m \longrightarrow \infty$, its strong ergodicity is assured.

On the other hand, if we write $P^{(m)} = P_1 \cdot P_2 \cdot P_3 \cdots P_m =$ $G + E_m$, where E_m is a matrix with each row sum equal to zero, then the strong ergodicity assumption implies that there exists a positive integer M such that given $\varepsilon > 0$, for $m \geq M$ $\|P_1 \cdot P_2 \cdot P_3 \cdots P_m - G\| < \varepsilon$; that is, $\|E_m\| < \varepsilon$. Therefore, letting $e_{m,i\ell}$ be the $(i,\ell)^{th}$ element of the matrix E_m ,

$$\delta(P^m) = \delta(G + E_m)$$

$$= \left(\frac{1}{2}\right) \sup_{i,j} \sum_{\ell=1}^{N} |(\pi_{i\ell} + e_{m,i\ell}) - (\pi_{j\ell} + e_{m,j\ell})|$$

$$= \left(\frac{1}{2}\right) \sup_{i,j} \sum_{\ell=1}^{N} |e_{m,i\ell} - e_{m,j\ell}|$$

$$\leq \left(\frac{1}{2}\right) \sup_{i,j} \sum_{\ell=1}^{N} \{|e_{m,i\ell}| + |e_{m,j\ell}|\}$$

$$< \varepsilon .$$

$\therefore$ The proof is complete.

We have so far studied on the convergence of general nonstationary Markov chains. In the real world, however, we may often see some regularly fluctuating demands (for example, seasonal fluctuation). This case can be easily treated independently of the previous theorems.

If a demand process $\{N_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ appears in a cyclic pattern, then the sequence of the transition matrices $\{P_k\}_{k=1}^{\infty}$ of the corresponding inventory position process $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$, for which subsequences converge even if the chain itself is not strongly ergodic, will have a trend such that the P_k's repeat themselves in a cyclic fashion; that is, $P_{nd+\ell} = P_\ell$ for $\ell = 1, 2, \ldots, d$ and $n = 0, 1, 2, \ldots,$ where $k \equiv nd + \ell$ and d is the number of system reviews within each repeating cycle. Therefore, we may be able to evaluate the limit distributions of subsequences of $\{IP_{T_{nd+\ell}}\}_{n=0}^{\infty}$ for $\ell = 1, 2, \ldots, d$. For example, if a demand pattern is seasonally fluctuated and so $d = 4$, then the limit distributions of subsequences of $\{IP_{T_1}, IP_{T_5}, IP_{T_9}, \ldots\}$, $\{IP_{T_2}, IP_{T_6}, IP_{T_{10}}, \ldots\}$, $\{IP_{T_3}, IP_{T_7}, IP_{T_{11}}, \ldots\}$ and

$\{\mathbb{P}_{T_4}, \mathbb{P}_{T_8}, \mathbb{P}_{T_{12}}, \ldots\}$ can be computed.

The next theorem, which has been proved by Bowerman, David and Isaacson (1977), summarizes the idea of how to compute the long-run distribution of the cyclic nonstationary Markov chain $\{\mathbb{P}_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$.

<u>Theorem III.B.8</u>: Let $\{P_k\}_{k=1}^{\infty}$ be a nonstationary Markov chain such that $P_{nd+\ell} = P_\ell$ for $\ell = 1, 2, \ldots, d$ and $n = 0, 1, 2, \ldots$ Assume that $R_d = P_1 \cdot P_2 \cdots P_d$ is strongly ergodic with constant matrix G_d. Moreover, let $G_\ell = G_d \prod_{i=1}^{\ell} P_i$ for $\ell = 1, 2, 3, \ldots, d-1$. Then,

if $R_1 = P_2 \cdot P_3 \cdots P_d \cdot P_1$, $R_2 = P_3 \cdot P_4 \cdots P_d \cdot P_1 \cdot P_2, \ldots,$ $R_{d-1} = P_d \cdot P_1 \cdot P_2 \cdots P_{d-1}$, there exist finite constants C and β $(0 < \beta < 1)$ such that for $n \geq 2$

a) $\|R_d^n - G_d\| \leq C\,\beta^n$

b) $\|R_\ell^n - G_\ell\| \leq C\,\beta^{n-1}$ for $\ell = 1, 2, \ldots, d-1$.

Proof:

Under the strong ergodicity assumption of R_d, (a) follows by Huang, Isaacson and Vinograde (1976).

The inequality (b) follows since for $\ell = 1, 2, \ldots, d-1$,

$$\|R_\ell^n - G_\ell\| = \|(\prod_{i=1}^{\ell} P_i)\,(R_\ell)^{n-1} - G_\ell\|, \text{ where } R_\ell = P_{\ell+1}\,P_{\ell+2} \cdots P_d\,P_1\,P_2 \cdots P_\ell$$

$$= \|(R_d)^{n-1} \prod_{i=1}^{\ell} P_i - G_d \prod_{i=1}^{\ell} P_i\|$$

$$\leq \|(R_d)^{n-1} - G_d\| , \quad \text{since} \quad \|\prod_{i=1}^{\ell} P_i\| = 1$$

$$\leq C \beta^{n-1} , \quad \text{for} \quad n \geq 2 .$$

$\therefore$ The proof is complete.

C. Limit Distribution of Inventory Position with Nonstationary Poisson Demand

As is pointed out earlier, the long-run limit distribution of a nonstationary Markov chain is the row vector of a constant stochastic matrix G which is the limit matrix of nonstationary Markov matrices $P^{(n)}$ in n steps as $n \to \infty$, if the chain is ergodic in the strong sense. Therefore, before trying to find such long-run distribution of nonstationary inventory position process $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$, the transition matrices of the process corresponding to the $<nQ, r, T>$ and $<R, r, T>$ inventory system operating models shall be constructed first under the assumptions made for this work in Chapter I, and then the determination on the ergodicity of the Markov matrices will follow.

Let an integer-valued process $\{N_t ; t \geq 0\}$ represent the number of demand occurrences by time t. The process is said to have nonstationary independent increments if the random variables $\{D_{(t_{n-1}, t_n]}\}$, defined in Section A of this chapter, are exclusively and mutually independent, and $\{D_{t_{n-1}+h, t_n+h]}\}$ for $h > 0$ does not have the same distribution as $\{D_{(t_{n-1}, t_n]}\}$ $(n = 1, 2, \ldots)$. When it is assumed that a process $\{X_t; t \geq 0\}$ is a family of exponentially distributed demand inter-arrival times and the corresponding counting process $\{N_t\}$ has nonstationary independent increments, the process $\{N_t\}$ turns out a nonstationary Poisson process. This process can be illustrated by use of the mean value function $E\{N_t\}$ denoted by $m(t)$. $m(t)$ is always assumed to be differentiable. Let λ and $\lambda(t)$ be, respectively, the mean rate at which counts are being made (or just intensity) and the derivative of $m(t)$ (or the intensity function), that is,

$$\lambda(t) = \frac{d}{dt} m(t) .$$

If the Poisson process $\{N_t; t \geq 0\}$ does not satisfy the condition that $m(t)$ is linearly proportional to t with proportionality factor λ, that is, $m(t) = E\{N_t\} = \lambda_t$, it is called a nonstationary Poisson process, or a Poisson process with nonstationary increments.

Thus, we need to know a precise form of $m(t)$ for the construction of transition matrices mentioned above. The approximate probability that at time $t \geq 0$, one Poisson event occurs within time increment Δt is $\lambda(t) \cdot \Delta t$. Let $(0, t]$ be a time interval such that there exists a large positive integral multiplier n satisfying $n \cdot \Delta t = t$, so that $\lambda(\ell \cdot \Delta t) \cdot \Delta t$ for $\ell = 0, 1, 2, \ldots, n-1$ represents the approximate probability that such an event occurs within the time interval $(\ell \cdot \Delta t, (\ell+1) \cdot \Delta t]$, or at time $\ell \cdot \Delta t$ such an event occurs within time increment Δt . Denote by $p_\ell(t)$ the probability that one Poisson event occurs just within the time interval $(\ell \cdot \Delta t, (\ell+1) \cdot \Delta t]$ over time range $(0, t]$, so that for $\ell = 0, 1, 2, \ldots, n-1$,

$$p_\ell(t) = [1 - \lambda(0) \cdot \Delta t]\,[1 - \lambda(\Delta t) \cdot \Delta t] \cdots [1 - \lambda((\ell-1) \cdot \Delta t)\Delta t]\,[\lambda(\ell \cdot \Delta t) \cdot \Delta t]$$

$$\cdot\, [1 - \lambda((\ell+1) \cdot \Delta t) \cdot \Delta t] \cdots [1 - \lambda((n-1) \cdot \Delta t) \cdot \Delta t].$$

Denote by $\theta(t)$ the probability that one Poisson event occurs anywhere over the time range $(0, t]$. Then, since possible time intervals for such event occurrences are not overlapped each other, $p_\ell(t)$ for all ℓ are the probabilities of disjoint random events. Therefore,

$$\theta(t) = \lim_{n \to \infty} \sum_{\ell=0}^{n-1} p_\ell(t)$$

or

$$= \lim_{\Delta t \to 0} \sum_{\ell=0}^{n-1} p_\ell(t) \, .$$

We shall show that as $\Delta t \to 0$ (or $n \to \infty$) , $\theta(t)$ tends to the Poisson distribution, from which the mean value function $m(t)$ is to be identified.

Define by

$$D(t) = [1-\lambda(0)\cdot\Delta t]\,[1-\lambda(\Delta t)\cdot\Delta t]\,[1-\lambda(2\cdot\Delta t)\cdot\Delta t]\;\cdots\;[1-\lambda((n-1)\cdot\Delta t)\cdot\Delta t]$$

$$= \prod_{\ell=0}^{n-1} [1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t] \;.$$

When each $p_\ell(t)$ is multiplied by the relevant unit value $1 \equiv [1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t]/[1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t]$, $p_\ell(t)$ is set equal to the product of $[\lambda(\ell\cdot\Delta t)\cdot\Delta t]/[1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t]$ and the common term $D(t)$; namely,

$$p_\ell(t) = D(t)\cdot\frac{[\lambda(\ell\cdot\Delta t)\cdot\Delta t]}{[1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t]} \;.$$

Therefore, letting $\theta(t) = \lim_{\Delta t \to \infty} \theta(t, \Delta t)$,

$$\theta(t, \Delta t) = \sum_{\ell=0}^{n-1} p_\ell(t)$$

$$= \sum_{\ell=0}^{n-1} D(t)\cdot\frac{[\lambda(\ell\cdot\Delta t)\cdot\Delta t]}{[1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t]}$$

$$= D(t)\cdot\sum_{\ell=0}^{n-1}\frac{[\lambda(\ell\cdot\Delta\tau)\cdot\Delta t]}{[1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t]} \;.$$

Let $\underline{\lambda} = \inf \lambda(x)$ and $\overline{\lambda} = \sup \lambda(x)$ for $0 \leq x \leq t$. Then,

$$D(t)\cdot\sum_{\ell=0}^{n-1} [\lambda(\ell\cdot\Delta t)\cdot\Delta t]\,[1-\underline{\lambda}\cdot\Delta t]^{-1} \leq \theta(t, \Delta t)$$

$$\leq D(t) \sum_{\ell=0}^{n-1} [\lambda(\ell \cdot \Delta t) \cdot \Delta t] \, [1 - \bar{\lambda} \cdot \Delta t]^{-1} .$$

Taking limit,

$$\{ \underline{\lim}_{\Delta t \to 0} D(t) \} \cdot \{ \underline{\lim}_{\Delta t \to 0} \sum_{\ell=0}^{n-1} [\lambda(\ell \cdot \Delta t) \cdot \Delta t] \} \cdot \{ \underline{\lim}_{\Delta t \to 0} [1 - \underline{\lambda} \cdot \Delta t]^{-1} \}$$

$$\leq \underline{\lim}_{\Delta t \to 0} \theta(t, \Delta t) , \qquad \text{(A)}$$

and $\overline{\lim}_{\Delta t \to 0} \theta(t, \Delta)$

$$\leq \{ \overline{\lim}_{\Delta t \to 0} D(t) \} \; \{ \overline{\lim}_{\Delta t \to 0} \sum_{\ell=0}^{n-1} [\lambda(\ell \cdot \Delta t) \cdot \Delta t] \} \cdot \{ \overline{\lim}_{\Delta t \to 0} [1 - \bar{\lambda} \cdot \Delta t]^{-1} \}, \qquad \text{(B)}$$

where $\underline{\lim}_{\Delta t \to 0}$ and $\overline{\lim}_{\Delta t \to 0}$ mean, respectively, $\lim\inf_{\Delta t \to 0}$ and $\lim\sup_{\Delta t \to 0}$.

Since

$$\lim_{\Delta t \to 0} \sum_{\ell=0}^{n-1} [\lambda(\ell \cdot \Delta t) \cdot \Delta t] = \int_0^t \lambda(x) \, dx , \qquad \text{and}$$

$$\lim_{\Delta t \to 0} [1 - \underline{\lambda} \cdot \Delta t]^{-1} = 1 = \lim_{\Delta t \to 0} [1 - \bar{\lambda} \cdot \Delta t]^{-1} , \text{ from (A) and (B) ,}$$

$$\overline{\lim}_{\Delta t \to 0} \theta(t, \Delta t) \leq \{ \lim_{\Delta t \to 0} D(t) \} \int_0^t \lambda(x) \, dx \leq \underline{\lim}_{\Delta t \to 0} \theta(t, \Delta t) .$$

Therefore,

$$\lim_{\Delta t \to 0} \Theta(t, \Delta t) = \{ \lim_{\Delta t \to 0} D(t)\} \int_0^t \lambda(x)\, dx .$$

Given $\epsilon > 0$, it is possible to choose δ such that for $\Delta t \leq \delta$ and $0 \leq x \leq t$, $\lambda(x)\cdot\Delta t < \epsilon$ and

$$e^{-1-\epsilon'} \leq [1-\lambda(x)\cdot\Delta t]^{\frac{1}{\lambda(x)\Delta t}} \leq e^{-1+\epsilon'} , \quad (\epsilon' \text{ is arbitrary}),$$

which is true from

$$[1-\lambda(x)\cdot\Delta t] = [(1-\lambda(x)\cdot\Delta t^{\frac{1}{\lambda(x)\Delta t}}]^{\lambda(x)\Delta t} \longrightarrow e^{-\lambda(x)\Delta t} ,$$

$$\text{as } \Delta t \longrightarrow 0 .$$

Thus

$$e^{(-1-\epsilon')\,[\lambda(x)\Delta t]} \leq [1 - \lambda(x)\cdot\Delta t] \leq e^{(-1+\epsilon')\,[\lambda(x)\Delta t]} ,$$

and so,

$$e^{(-1-\epsilon')\sum_{\ell=0}^{n-1} \lambda(\ell\cdot\Delta t)\cdot\Delta t} \leq \prod_{\ell=0}^{n-1} [1-\lambda(\ell\cdot\Delta t)\cdot\Delta t]$$

$$\leq e^{(-1+\epsilon')\sum_{\ell=0}^{n-1} \lambda(\ell\cdot\Delta t)\cdot\Delta t} , \qquad (C)$$

$$\therefore \quad \lim_{\Delta t \to 0} D(t) = \lim_{\Delta t \to 0} \prod_{\ell=0}^{n-1} [1 - \lambda(\ell\cdot\Delta t)\cdot\Delta t$$

$$= e^{-\int_0^t \lambda(x)dx} , \quad \text{from } (C) .$$

$$\therefore \quad \theta(t) = \lim_{\Delta t \to 0} \theta(t, \Delta t)$$

$$= e^{-\int_0^t \lambda(x)dx} \cdot \int_0^t \lambda(x)dx \; ,$$

which shows the mean value function

$$m(t) = E\{N_t\} = \int_0^t \lambda(x)\, dx \; . \tag{3.3.1}$$

Let $\{T_k;\ k = 0, 1, 2, \dots\}$ be a sequence of time intervals corresponding to an inventory system review periods with $T_0 \equiv 0$. Then, the mean value function $m(I_{k+1})$, where I_{k+1} represents a time interval $(T_k, T_{k+1}]$ corresponding to a Poisson demand counting process $D_{(T_k, T_{k+1}]}$ during the $(k+1)^{st}$ review period, is

$$m(I_k) = E\{D_{(T_k, T_{k+1}]}\}$$

$$= E\{N_{T_{k+1}} - N_{T_k}\}$$

$$= \int_0^{T_{k+1}} \lambda(x)dx - \int_0^{T_k} \lambda(x)dx \; , \quad \text{from Eq. (3.3.1)} \; ,$$

$$= \int_{T_k}^{T_{k+1}} \lambda(x)dx \quad , \tag{3.3.2}$$

$$\therefore \quad P\{D_{(T_k,T_{k+1}]}\} = P\{N_{T_{k+1}} - N_{T_k} = n\}$$

$$= \frac{e^{-\int_{T_k}^{T_{k+1}} \lambda(x)dx} \cdot \left(\int_{T_k}^{T_{k+1}} \lambda(x)dx\right)^n}{n!} . \quad (3.3.3)$$

It will be proved in the next section that $\{IP_{T_k}\}$ and $\{D_{(T_k,T_k+h]};\ h \geq 0\}$ $(k = 0, 1, 2, \ldots)$ are mutually independent of each other. Then, it follows from Eq. (3.3.3) that for $\tau \leq \xi \leq \tau + \Delta T_k$ $(\Delta T_k \equiv T_{k+1} - T_k)$ and $j = 1, 2, \ldots, Q$,

$$P\{IP_{T_k} = r+j,\ D_{(T_k,T_k+\xi]} = m\} = P\{IP_{T_k} = r+j,\ N_{T_k+\xi} - N_{T_k} = n\}$$

$$= P\{IP_{T_k} = r+j\}\ P\{N_{T_k+\xi} - N_{T_k} = n\}$$

$$= P\{IP_{T_k} = r+j\} \cdot \frac{e^{-\int_{T_k}^{T_k+\xi} \lambda(x)dx} \cdot \left[\int_{T_k}^{T_k+\xi} \lambda(x)dx\right]^n}{n!} . \quad (3.3.4)$$

Since $NIS_{T_k+\xi} = IP_{T_k} - D_{(T_k,T_k+\xi]}$ $(k = 0, 1, 2, \ldots)$, if the asymptotic limit distribution of IP_{T_k} is known, the asymptotic limit distribution of $NIS_{T_k+\xi}$ can be computed by the relation in Eq. (3.3.4), so that finally the expected average annual cost analysis can be derived. In fact, the Markov chain theory discussed in the preceding section can be directly applied to solve for the asymptotic distribution of $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$.

Using the independence assumption on demands during different review periods and also in view of the fact that $\{IP_{T_k}\}$ and $\{D_{(T_k,T_k+h]};\ h \geq 0\}$ $(k = 0, 1, 2, \ldots)$ are mutually independent of each other, we shall show that the intensity function $\lambda(t)$ of nonhomogeneous Poisson demand process yields the parameters of the nonstationary Markov chain process for inventory position under $<nQ, r, T>$ and $<R, r, T>$ models. It is sufficient to show that the sequence of demands, $D_{(T_k,T_{k+1}]}$, during the corresponding review period $\{I_{k+1}\}$ directly affects the next relations to follow:

(A) $$P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_k} = i_k,\ IP_{T_{k-1}} = i_{k-1},\ \ldots,\ IP_{T_0} = i_0\}$$

$$= P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_k} = i_k\} ,$$

for $i_k \in S = \{r+1,\ r+2,\ \ldots,\ r+Q\}$ or $S = \{r+1,\ r+2, \ldots, R\}$, $\forall\ k$,

(B) $$P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_k} = i_k\} \neq P\{IP_{T_{k+1}+h} = i_{k+1} \mid IP_{T_k+h} = i_k\} ,$$

for $h > 0$.

In order to show the equality of (A), we can use the relation,

$$P\{IP_{T_{k-1}} = i_{k-1},\ IP_{T_k} = i_k,\ IP_{T_{k+1}} = i_{k+1}\} = P\{IP_{T_{k-1}} = i_{k-1}\}$$

$$\cdot\ P\{IP_{T_k} = i_k \mid IP_{T_{k-1}} = i_{k-1}\} \cdot P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_{k-1}} = i_{k-1},\ IP_{T_k} = i_k\},$$

as follows;

$$P\{IP_{T_{k-1}} = i_{k-1},\ IP_{T_k} = i_k,\ IP_{T_{k+1}} = i_{k+1}\}$$

$$= P\{IP_{T_{k-1}} = i_{k-1}\} \cdot P\{IP_{T_k} = i_k,\ IP_{T_{k+1}} = i_{k+1} \mid IP_{T_{k-1}} = i_{k-1}\}$$

$$= P\{IP_{T_{k-1}} = i_{k-1}\} \cdot P\{D_{(T_{k-1},T_k]} = f(i_{k-1},\ i_k),\ D_{(T_k,T_{k+1}]} = f(i_k,\ i_{k+1}) \mid IP_{T_{k-1}} = i_{k-1}\}$$

$$= P\{IP_{T_{k-1}} = i_{k-1}\} \cdot P\{D_{(T_{k-1},T_k]} = f(i_{k-1}, i_k)\} \cdot P\{D_{(T_k,T_{k+1}]} = f(i_k,\ i_{k+1})\},$$

where $f(i_k,\ i_{k+1})$ is the functional measure for demands during the $(k+1)^{st}$ period which get $IP_{T_{k+1}}$ equal to i_{k+1} given $IP_{T_k} = i_k$, and

$$P\{IP_{T_k} = i_k \mid IP_{T_{k-1}} = i_{k-1}\} = P\{D_{(T_{k-1},T_k]} = f(i_{k-1}, i_k) \mid IP_{T_{k-1}} = i_{k-1}\}$$

$$= P\{D_{(T_{k-1},T_k]} = f(i_{k-1}, i_k)\} ,$$

whence

$$P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_{k-1}} = i_{k-1}, IP_{T_k} = i_k\}$$

$$= \frac{P\{IP_{T_{k-1}} = i_{k-1}, IP_{T_k} = i_k, IP_{T_{k+1}} = i_{k+1}\}}{P\{IP_{T_{k-1}} = i_{k-1}\} \cdot P\{IP_{T_k} = i_k \mid IP_{T_{k-1}} = i_{k-1}\}}$$

$$= P\{D_{(T_k,T_{k+1}]} = f(i_k, i_{k+1})\}$$

$$= P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_k} = i_k\} .$$

Similarly, $P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_k} = i_k, IP_{T_{k-1}} = i_{k-1}, IP_{T_{k-2}} = i_{k-2}\}$ $= P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_k} = i_k\}$, and so on.

For the relation of (B), without loss of generality, we consider the case, $i_k \geq i_{k+1}$, in which the demand $D_{(T_k,T_{k+1}]} = i_k - i_{k+1}$ is needed for the transition of inventory position of going from state i_k to state i_{k+1} . Then

$$P\{IP_{T_{k+1}} = i_{k+1} \mid IP_{T_k} = i_k\} = P\{D_{(T_k,T_{k+1}]} = i_k - i_{k+1} \mid IP_{T_k} = i_k\}$$

$$= P\{D_{(T_k, T_{k+1}]} = i_k - i_{k+1}\}$$

$$= \frac{e^{-\int_{T_k}^{T_{k+1}} \lambda(x)dx} \cdot \left(\int_{T_k}^{T_{k+1}} \lambda(x)dx\right)^{i_k - i_{k+1}}}{(i_k - i_{k+1})!} ,$$

from Eq. (3.3.3) .

Likewise,

$$P\{IP_{T_{k+1}+h} = i_{k+1} \mid IP_{T_k+h} = i_k\} = P\{D_{(T_k+h, T_{k+1}+h]} = i_k - i_{k+1}\}$$

$$= \frac{e^{-\int_{T_k+h}^{T_{k+1}+h} \lambda(x)dx} \cdot \left(\int_{T_k+h}^{T_{k+1}+h} \lambda(x)dx\right)^{i_k - i_{k+1}}}{(i_k - i_{k+1})!} .$$

However, since $\{N_{T_k}\}$ is assumed a Poisson process with nonstationary independent increments, $\lambda(t)$ is not linearly proportional to t , with proportionality factor λ . Therefore,

$$\int_{T_k}^{T_{k+1}} \lambda(x)dx \neq \int_{T_k+h}^{T_{k+1}+h} \lambda(x)dx .$$

Hence, (A) and (B) hold, so that the process $\{IP_{T_k} ; T_k \geq 0$ for $k = 0, 1, 2, \ldots\}$ is a nonstationary Markov chain.

We shall first compute the transition probabilities of the non-stationary finite Markov chain $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ $(T_0 \equiv 0)$ with the finite state space $S' = \{r+1, r+2, \ldots, r+Q\}$ or $S = \{1, 2, \ldots, Q\}$ for the $<nQ, r, T>$ inventory system operating model.

Recall that under the $<nQ, r, T>$ model, an order is placed at a review time T_k $(k = 0, 1, 2, \ldots)$ if and only if the inventory position $\{IP_{T_k}\}$ of the system is less than or equal to r. If $IP_{T_k} \leq r$, then a quantity nQ $(n = 1, 2, 3, \ldots)$ is ordered, where n is chosen such that $r < IP_{T_k} + nQ \leq r+Q$. Therefore, immediately after a review, the inventory position will be in one of the Q states $r+1$, $r+2, \ldots, Q$.

Denote by $p_{k,ij}$ the transition probability of going from state i to state j in the $(k+1)^{st}$ step (or during the $(k+1)^{st}$ review period $(T_k, T_{k+1}]$) given that the process was in state i at time T_k, namely

$$p_{k,ij} = P\{IP_{T_{k+1}} = r+j \mid IP_{T_k} = r+i\} \quad \text{for } k = 0, 1, 2, \ldots, \text{ and all } i,j \in S .$$

If $j \leq i$, this transition probability exists only when $\{\ell\cdot Q + (i-j)\}$ $(\ell = 0, 1, 2, \ldots)$ units have been demanded in the interval $(T_k, T_{k+1}]$, that is, $D_{(T_k,T_{k+1}]} = \ell\cdot Q + (i-j)$ for all $i,j \in S$. On the other hand, if $j > i$, the transition probability exists only when

$D_{(T_k,T_{k+1}]} = \ell \cdot Q + (i-j)$ ($\ell = 1, 2, \ldots$; all $i,j \in S$). Otherwise, the probability is zero.

Based on the above two different kinds of demand impact on inventory positions, the corresponding transition matrix P_k, say, is constructed in Table III.1. For notational convenience, denote by $\phi_{k,i}$ the probabilities of $\{i + \ell \cdot Q\}$ units demand occurrences during the $(k+1)^{st}$ review period for $\ell = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots, Q-1$, namely,

$$\sum_{\ell=0}^{\infty} P\{D_{(T_k,T_{k+1}]} = \ell \cdot Q + i\} = \phi_{k,i}, \quad \text{for } i = 0, 1, 2, \ldots, Q-1 .$$

$\{\phi_{k,i}\}$ is notified in Table III.1. So, it is easy to determine that the transition probability matrix is doubly stochastic. The matrix is composed of all positive entries. Therefore, the matrix is primitive.

Hence, if the nonstationary Markov chain $\{IP_{T_k}; T_k \geq 0\}_{k=0}^{\infty}$ is weakly ergodic, then it follows from Theorem III.B.7 that there exist a limit constant matrix G, each row vector π of which is uniform; that is,

$$\begin{aligned} \pi &= (\pi_{r+1}, \pi_{r+2}, \ldots, \pi_{r+Q}) \\ &= \left(\frac{1}{Q}, \frac{1}{Q}, \ldots, \frac{1}{Q} \right) . \end{aligned} \tag{3.3.5}$$

If we consider the chain $\{IP_{T_k}\}_{k=0}^{\infty}$ associated with the nonstationary (or nonhomogeneous) Poisson demand process $\{D_{(T_k,T_{k+1}]}\}$, in

Table III.1. Transition matrix of $\{IP_{T_k}\}$ for $<nQ, r, T>$ model

IP_{T_k} \ $IP_{T_{k+1}}$	$\{r+j\}$ $(j = 1, 2, \ldots, Q)$					
	$r+1$	$r+2$	$r+3$	....	$r+Q-1$	$r+Q$
$r+1$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q\} = \phi_{k,0}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-1\} = \phi_{k,Q-1}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-2\} = \phi_{k,Q-2}$	....	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+2\} = \phi_{k,2}$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+1\} = \phi_{k,1}$
$r+2$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+1\} = \phi_{k,1}$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q\} = \phi_{k,0}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-1\} = \phi_{k,Q-1}$	....	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+3\} = \phi_{k,3}$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+2\} = \phi_{k,2}$
$r+3$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+2\} = \phi_{k,2}$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+1\} = \phi_{k,1}$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q\} = \phi_{k,0}$	....	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+4\} = \phi_{k,4}$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+3\} = \phi_{k,3}$
$\{r+i\}$	....	....	....	....	....	....
$r+Q-1$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-2\} = \phi_{k,Q-2}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-3\} = \phi_{k,Q-3}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-4\} = \phi_{k,Q-4}$	....	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q\} = \phi_{k,0}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-1\} = \phi_{k,Q-1}$
$r+Q$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-1\} = \phi_{k,Q-1}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-2\} = \phi_{k,Q-2}$	$\sum_{\ell=1}^{\infty} P\{D_{I_k} = \ell \cdot Q-3\} = \phi_{k,Q-3}$	....	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q+1\} = \phi_{k,1}$	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell \cdot Q\} = \phi_{k,0}$

Note: $I_k = (T_k, T_{k+1}]$

which the transition probability matrices, denoted by $\{P_k\}$, repeat themselves in a cyclic fashion such as $P_{nd+\ell} = P_\ell$ ($\ell = 1, 2, \ldots, d$; $n = 0, 1, 2, \ldots$), then since the P_k's are primitive it can be verified in view of Theorem III.B.8, or Corollary III.B.1 and Theorem III.B.7 that the long-run limit distribution of $\{IP_{T_k}\}$, denoted by π_c, is also uniform; that is,

$$\pi_c = \left(\frac{1}{Q}, \frac{1}{Q}, \ldots, \frac{1}{Q} \right) . \tag{3.3.6}$$

By the direct use of Theorem III.B.8, we can get the above result, since $\{R_d = \prod_{\ell=1}^{d} P_\ell\}$ is strongly ergodic and the constant matrices $\{G_\ell;\ \ell = 1, 2, \ldots, d\}$ are composed of all the same row vectors $\pi = \left(\frac{1}{Q}, \frac{1}{Q}, \ldots, \frac{1}{Q} \right)$ under the $\langle nQ, r, T \rangle$ model.

In order to apply Corollary III.B.1, denote by σ_j^k a least element of the j^{th} column of P_k for $j = 1, 2, \ldots, Q$. Then, from Table III.1

$$\sigma_j^k = \min_{i \in I} \left[\sum_{m=0}^{\infty} P\,\{D_{(T_k, T_{k+1}]} = mQ + i\} \right]$$

$$= \min_{i \in I} \{\phi_{k,i}\} \quad \text{over the set } I = \{0, 1, 2, \ldots, Q-1\},$$
$$\text{for } j = 1, 2, \ldots, Q .$$

Therefore,

$$\sigma^k = \max_{j \in J} \{\sigma_j^k\} \quad \text{over} \quad J = \{1, 2, \ldots, Q\}$$

$$= \max_{j \in J} \left[\min_{i \in I} \{\phi_{k,i}\} \right]$$

$$= \min_{i \in I} \{\phi_{k,i}\} \ .$$

Thus, $\sigma^k \geq \min_{i \in I} P\{D_{(T_k, T_{k+1}]} = i\}$ over $I = \{0, 1, 2, \ldots, Q-1\}$,

for $k = 0, 1, 2, \ldots$

Moreover, since $P\{D_{(T_k, T_{k+1}]} = i\} = \dfrac{e^{-\int_{T_k}^{T_{k+1}} \lambda(x)dx} \cdot \left[\int_{T_k}^{T_{k+1}} \lambda(x)dx\right]^i}{i!}$,

there exists a value $\sigma > 0$ such that

$$\sigma = \min_{i \in I} P\{D_{(T_k, T_{k+1}]} = i\}$$

$$= \min_{i \in I} P\{D_{(T_{nd+\ell}, T_{nd+\ell+1}]} = i\} ,$$

where $k = nd + \ell$ $(\ell = 1, 2, \ldots, d;\ n = 0, 1, 2, \ldots)$,

since P_k's repeat themselves,

$$\geq \min_{\substack{i \in I \\ \ell \in L}} P\{D_{(T_\ell, T_{\ell+1}]} = i\} > 0 \quad \text{over} \quad L = \{1, 2, \ldots, d\}$$

$$\therefore \quad \sigma^k \geq \sigma > 0 \ .$$

Therefore, the chain $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ is weakly ergodic referring to Corollary III.B.1, and the uniformity result π_c follows by Theorem III.B.7.

Under the stationary finite Markov chain assumption of the process $\{IP_{kT}; T > 0$ and $k = 0, 1, 2, \ldots\}$, where T is the constant system review period, the next equality follows;

$$P\{IP_{(k+1)T} = r + j \mid IP_{kT} = r + i\} = P\{IP_{2T} = r + j \mid IP_T = r + i\}$$

$$(k = 1, 2, \ldots), \quad \text{for all } \; i,j \in S \; .$$

When we let p_{ij} represent the transition probability of going from state i to state j , Corollary III.B.2 is directly applied to get the long-run limit distribution $\pi = (\frac{1}{Q}, \frac{1}{Q}, \ldots, \frac{1}{Q} \}$, since the transition probability matrix is primitive.

Hence, we conclude that Theorem III.B.7 is robust for the $<nQ, r, T>$ model, because whatever the demand distributions are they will be formed into the corresponding doubly stochastic matrices for $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ and thence the uniform limit distribution will be achieved.

Now, we shall study on the transition probabilities of the non-stationary finite Markov chain $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ with the state space $S^* = \{1, 2, \ldots, R - r\}$ under the $<R, r, T>$ model.

Recall that under the $<R, r, T>$ model, if the inventory position $\{IP_{T_k}\}$ at a review time T_k $(k = 0, 1, 2, \ldots)$ is less than or equal to r , then an order is placed immediately after the review time to bring the inventory position up to R .

Define by $p_{k,ij} = P\{IP_{T_{k+1}} = r+j \mid IP_{T_k} = r+i\}$ for all $i,j \in S^*$ and $k = 0, 1, 2, \ldots$ Then, the positive transition probability exists only when at least one of the following conditions is satisfied:

(a) For $j \leq i$ and $j \neq R-r$, $D_{(T_k,T_{k+1}]} = i-j$ for all $i,j \in S^*$.

(b) For $i \neq R-r$ and $j = R-r$, $D_{(T_k,T_{k+1}]} = \ell+i$ for all $i \in S^*$ and $\ell = 0, 1, 2, \ldots$

(c) For $i = j = R-r$, $D_{(T_k,T_{k+1}]} = \{0\}$

$= \{\ell + (R-r)\}$

for $\ell = 0, 1, 2, \ldots$

Otherwise, the transition probability is zero. The corresponding transition probability matrix P_k, composed of the components $\{p_{k,ij}\}$, is established in Table III.2. As was done for the <nQ, r, T> model, let $\theta_{k,i}$ represent the probability of i demand occurrences during the $(k+1)^{st}$ review period; or formally,

$$\theta_{k,i} = P\{D_{(T_k,T_{k+1}]} = i\} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

$\{\theta_{k,i}\}$ are shown in Table III.2.

We shall investigate the transition probability matrices $\{P_k\}$ concerning primitivity and ergodicity (existence of a long-run limit distribution). The matrix of Table III.2 shows that column "R", column "r + 1" and the first two top rows are composed of all positive

Table III.2. Transition matrix of $\{IP_{T_k}\}$ for $\langle R, r, T \rangle$ model

IP_{T_k} \ $IP_{T_{k+1}}$		$\{r+j\}$ $(j = 1, 2, \ldots, Q)$					
		R	R - 1	R - 2	....	r + 2	r + 1
	R	$P\{D_{I_k} = 0\} + \sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell+(R-r)\} = \theta_0 + \sum_{\ell=1}^{\infty} \theta_{\ell+R-r}$	$P\{D_{I_k} = 1\} = \theta_1$	$P\{D_{I_k} = 2\} = \theta_2$	....	$P\{D_{I_k} = R-2-r\} = \theta_{R-2-r}$	$P\{D_{I_k} = R-1-r\} = \theta_{R-1-r}$
	R-1	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell+(R-1-r)\} = \sum_{\ell=0}^{\infty} \theta_{\ell+(R-1-r)}$	$P\{D_{I_k} = 0\} = \theta_0$	$P\{D_{I_k} = 1\} = \theta_1$	....	$P\{D_{I_k} = R-3-r\} = \theta_{R-3-r}$	$P\{D_{I_k} = R-2-r\} = \theta_{R-2-r}$
$\{r+i\}$	R-2	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell+(R-2-r)\} = \sum_{\ell=0}^{\infty} \theta_{\ell+(R-2-r)}$	0	$P\{D_{I_k} = 0\} = \theta_0$	....	$P\{D_{I_k} = R-4-r\} = \theta_{R-4-r}$	$P\{D_{I_k} = R-3-r\} = \theta_{R-3-r}$
	...	....	....	....	....	....	....
	r+1	$\sum_{\ell=0}^{\infty} P\{D_{I_k} = \ell+1\} = \sum_{\ell=0}^{\infty} \theta_{\ell+1}$	0	0	....	0	$P\{D_{I_k} = 0\} = \theta_0$

Note: $I_k = (T_k, T_{k+1}]$

entries, and the diagonal elements are all positive, too. Therefore, it is easily determined that each transition matrix P_k is primitive.

Moreoever, Theorem III.B.6 insures that each transition matrix $\{P_k\}$ has the unique left eigenvector $\{\pi_k\}$ corresponding to eigenvalue 1. Therefore, if the nonstationary Markov chain $\{\mathbb{P}_{T_k}; T_k \geq 0\}_{k=0}^{\infty}$ under the <R, r, T> model satisfies the weak ergodicity in Theorem III.B.4, then by Theorem III.B.5 the long-run limit distribution of $\{\mathbb{P}_{T_k}\}_{k=0}^{\infty}$ can be computed; namely,

$$\lim_{k \to \infty} \|\pi_k - \pi\| = 0 .$$

The possible cyclic behaviour of demand patterns can also be taken into account for this <R, r, T> case. For example, a seasonal trend of Poisson demand process $\{N_t; t \geq 0\}$ may affect the $\{\mathbb{P}_{T_k}; T_k \leq 0\}_{k=0}^{\infty}$ to be kept in the seasonal fashion with $d = 4$.

From the transition probability matrices P_i's shown in Table III.2, it is easily checked that $R_4 = P_1 \cdot P_2 \cdot P_3 \cdot P_4$ is composed of all positive entries. Recall that a stationary finite Markov chain is strongly ergodic if and only if it is weakly ergodic. Therefore, the stationary finite transition matrix R_4 is strongly ergodic, so that it will converge to a constant matrix, G_4 (say). Hence, the application of Theorem III.B.8 and Theorem III.B.6 to these problems of cyclic demand patterns will determine the constant matrices $\{G_\ell; \ell = 1, 2, 3, 4\}$ denoting the seasonal long-run limit distribution of

$\{IP_{T_k}; T_k \geq \}_{k=0}^{\infty}$ (or $\{IP_{T_{nd+\ell}}; T_{nd+\ell} \geq 0,\ \ell = 1, 2, \ldots, d$ and $n = 0, 1, 2, \ldots\}$).

The above claim can be illustrated with the case of the convergence of the inventory positions at the end of every second season, in which the sequence of $\{P_{T_2}, T_{T_6}, T_{T_{10}}, \ldots, T_{T_{4n+2}}, \ldots\}$ converges to G_2 (say) as $n \rightarrow \infty$. Denote by $S_2^{(n)}$ the transition probability matrix of the $\{IP_{T_{nd+2}}; d = 4\}$ for the second season in the n^{th} year. Then

$$S_2^{(n)} = (P_1 \cdot P_2)\ \underbrace{(P_3 \cdot P_4 \cdot P_1 \cdot P_2)\ (P_3 \cdot P_4 \cdot P_1 \cdot P_2) \cdots (P_3 \cdot P_4 \cdot P_1 \cdot P_2)}_{(n-1)\ \text{repetitions}}$$

$$= (P_1 \cdot P_2 \cdot P_3 \cdot P_4)^{n-1}\ (P_1 \cdot P_2)$$

$$= (R_4)^{n-1}\ (P_1 \cdot P_2)\ .$$

Therefore, if $(R_4)^n$ converges to G_4 as $n \rightarrow \infty$, then $S_2^{(n)}$ converges to $G_2 = G_4 \cdot (P_1 \cdot P_2)$ as $n \rightarrow \infty$, which means that the convergence follows by Theorem III.B.8.

As was done under the $<nQ, r, T>$ model, the following investigation verifies that Corollary III.B.1 can also be used to determine the weak ergodicity of the chain $\{IP_{T_k}; T_k \geq 0\}_{k=0}^{\infty}$ under this $<R, r, T>$ model, since one of the entries in column "R" will become the positive maximum value among entries chosen as the least element from each column. Therefore,

$$\sigma^k = \min_{i \in I} [P\{D_{(T_k, T_{k+1}]} = 0\} + \sum_{m=0}^{\infty} P\{D_{(T_k, T_{k+1}]} = m + (R-r)\} ,$$

$$\sum_{m=0}^{\infty} P\{D_{(T_k, T_{k+1}]} = m + i\}] ,$$

over the set $I = \{0, 1, 2, \ldots, R-3-r, R-2-r, R-1-r\}$

$$= \min_{i \in I} [P\{D_{(T_{nd+\ell}, T_{nd+\ell+1}]} = 0\} + \sum_{m=0}^{\infty} P\{D_{(T_{nd+\ell}, T_{nd+\ell+1}]} + m + (R-r)\},$$

$$\sum_{m=0}^{\infty} P\{D_{(T_{nd+\ell}, T_{nd+\ell+1}]} = m + i\}] ,$$

where $k = nd + \ell$ $(\ell = 1, 2, \ldots, d; \; n = 0, 1, 2, \ldots)$

$$\geq \min_{\substack{i \in I \\ \ell \in L}} [P\{D_{(T_\ell, T_{\ell+1}]} = 0\} + \sum_{m=0}^{\infty} P\{D_{(T_\ell, T_{\ell+1}]} = m + (R-r)\} ,$$

$$\sum_{m=0}^{\infty} P\{D_{(T_\ell, T_{\ell+1}]} = m + i\}] ,$$

over $L = \{1, 2, 3, \ldots, d\}$

$$= \sigma \text{ (say)} . \tag{3.3.7}$$

Hence, it follows by Corollary III.B.1 that the nonstationary Markov chain $\{IP_{T_{nd+\ell}} ; T_{nd+\ell} \geq 0; \ell = 1, 2, \ldots, d\}_{n=0}^{\infty}$ is weakly ergodic, since $\sigma^k \geq \sigma > 0$ shown in Eq. (3.3.7).

In the case of the stationary finite Markov chain $\{IP_{kT};\ T \geq 0$ and $k = 0, 1, 2, \ldots\}$ under the <R, r, T> model, the long-run limit distribution can be computed directly from the corresponding transition probability matrix P_T, say, consisted of $\{P_{T,ij}\}$ for all $i,j \in S^*$. $p_{T,ij}$ is defined to be

$$p_{T,ij} = P\{IP_{(k+1)T} = r+j \mid IP_{kT} = r+i\}$$

$$= P\{IP_{2T} = r+j \mid IP_T = r+i\} \quad \text{for all } i,j \in S^* \text{ and } k = 0, 1, 2, \ldots$$

In fact, the nonstationary finite transition matrix $\{P_k\}_{k=0}^{\infty}$ in Table III.2 can be considered as P_T by fixing the time index invariant, where T is a constant review period. For example, $\{D_{(kT,(k+1)T]} = i-j\}$, instead of $\{D_{(T_k,T_{k+1}]} = i-j\}$, will denote the $(i-j)$ demands during the $(k+1)^{st}$ review period. It was discussed earlier that the stationary transition matrix P_T is primitive and hence strongly ergodic, since a stationary finite Markov chain is strongly ergodic if and only if it is weakly ergodic. Thus, there exists the unique long-run limit distribution $\pi = \{\pi_{r+1}, \pi_{r+2}, \ldots, \pi_R\}$ of $\{IP_{kT};\ T \geq 0\}_{k=0}^{\infty}$ or the left eigenvector π for the matrix P_T corresponding to the eigenvalue 1. The same result also follows directly by Theorem III.B.6.

By the way, the system of equations $\pi P_T = \pi$ has infinitely many solutions. Therefore, the second condition $\sum_{i=r+1}^{R} \pi_i = 1$ must be applied to solve for the unique long-run limit distribution π.

In view of the matrix in Table III.2, it will be easier for us to solve the system of $(R - r)$ equations $\pi P_T^* = \pi$ and $\sum_{i=r+1}^{R} \pi_i = 1$, where P_T^* is the $(R-r) \times (R-r-1)$ matrix reduced by removing the column "R". With this reduced system of equations, we shall determine the closed form of solution vector π. Following is the system of equations the computations will start with;

$$\left.\begin{aligned}
&\sum_{i=r+1}^{R} \pi_i = 1 \quad \text{and from Table III.2 ,} \\
&\theta_1 \cdot \pi_R + \theta_0 \cdot \pi_{R-1} = \pi_{R-1} \\
&\theta_2 \cdot \pi_R + \theta_1 \cdot \pi_{R-1} + \theta_0 \cdot \pi_{R-2} = \pi_{R-2} \\
&\theta_3 \cdot \pi_R + \theta_2 \cdot \pi_{R-1} + \theta_1 \cdot \pi_{R-2} + \theta_2 \cdot \pi_{R-3} = \pi_{R-3} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\theta_{R-2-r} \cdot \pi_R + \theta_{R-3-r} \cdot \pi_{R-1} + \theta_{R-4-r} \cdot \pi_{R-2} + \cdots + \theta_1 \cdot \pi_{R+3} + \theta_0 \cdot \pi_{r+2} = \pi_{r+2} \\
&\theta_{R-1-r} \cdot \pi_R + \theta_{R-2-r} \cdot \pi_{R-1} + \theta_{R-3-r} \cdot \pi_{RO2} + \cdots + \theta_2 \cdot \pi_{R+3} + \theta_1 \cdot \pi_{r+2} \\
&\qquad + \theta_0 \cdot \pi_{r+1} = \pi_{r+1} .
\end{aligned}\right\} \tag{3.3.8}$$

Solving for π_{R-i} $(i = 1, 2, \ldots, R-1-r)$ in terms of π_R, we can get the following set of equations for which the coefficients have a recurrence relation;

$$\left.\begin{aligned}
(1-\theta_0)\cdot\pi_{R-1} &= \theta_1\cdot\pi_R \\
(1-\theta_0)\cdot\pi_{R-2} &= \theta_2\cdot\pi_R + \theta_1\cdot\pi_{R-1} = (\theta_1\cdot\frac{\theta_1}{1-\theta_0} + \theta_2)\cdot\pi_R \\
(1-\theta_0)\cdot\pi_{R-3} &= \theta_3\cdot\pi_R + \theta_2\cdot\pi_{R-1} + \theta_1\cdot\pi_{R-2} \\
&= \{\theta_1\cdot(\frac{1}{1-\theta_0})(\frac{\theta_1^2}{1-\theta_0} + \theta_2) + \theta_2\cdot(\frac{\theta_1}{1-\theta_0}) + \theta_3\}\cdot\pi_R
\end{aligned}\right\} \tag{3.3.9}$$

$$\left.\begin{aligned}
(1-\theta_0)\cdot\pi_{R-4} &= \theta_4\cdot\pi_R + \theta_3\cdot\pi_{R-1} + \theta_2\cdot\pi_{R-2} + \theta_1\cdot\pi_{R-3} \\
&= [\theta_1\cdot(\frac{1}{1-\theta_0})\{(\frac{\theta_1}{1-\theta_0})(\frac{\theta_1^2}{1-\theta_0} + \theta_2) + (\frac{\theta_1\cdot\theta_2}{1-\theta_0}) + \theta_3\}]\pi_R \\
&\quad + \theta_2\cdot\{(\frac{\theta_1^2}{1-\theta_0}) + \theta_2\} + \theta_3\cdot(\frac{\theta_1}{1-\theta_0}) + \theta_4
\end{aligned}\right\}$$

and so forth.

Let K_i denote the coefficients of π_R in equation π_{R-i} $(i = 1, 2, \ldots, R-1-r)$. Then, Eq. (3.3.9) can be simplified as follows:

$$(1-\theta_0)\,K_1 = \theta_1$$

$$(1-\theta_0)\,K_2 = \theta_1\cdot K_1 + \theta_2$$

$$(1-\theta_0)\,K_3 = \theta_1\cdot K_2 + \theta_2\cdot K_1 + \theta_3$$

$$(1-\theta_0)\,K_4 = \theta_1\cdot K_3 + \theta_2\cdot K_2 + \theta_3\cdot K_1 + \theta_4$$

$$\vdots$$

$$(1 - \theta_0) \cdot K_i = \sum_{j=1}^{i} \theta_j \cdot K_{i-j} \quad , \qquad (K_0 \equiv 1) \quad .$$

Thus

$$K_i = \frac{1}{1-\theta_0} \cdot \sum_{j=1}^{i} \theta_j \cdot K_{i-j} \ , \qquad (K_0 \equiv 1) \qquad (3.3.10)$$

$$\text{for} \quad i = 1, 2, \ldots, R\text{-}1\text{-}r \ .$$

Thence,

$$1 = \sum_{i=r+1}^{R} \pi_i$$

$$= \{(\sum_{i=r+1}^{R-1} K_i) + 1\} \, \pi_R \ .$$

Hence,

$$\left.\begin{array}{l} \pi_R = \dfrac{1}{1 + \sum_{i=r+1}^{R-1} K_i} \\ \\ \pi_{R-i} = K_i \cdot \pi_R \qquad (i = 1, 2, \ldots, R\text{-}1\text{-}r) \end{array}\right\} \ . \qquad (3.3.11)$$

These π_{R-i} $(i = 0, 1, 2, \ldots, R\text{-}1\text{-}r)$ in Eq. (3.3.11) can be easily computed on a Digital Computer once the probabilities θ_i's $(i = 0, 1, 2, \ldots, R\text{-}1\text{-}r)$ are known. Another approach to general $n \times n$ matrix problems has been suggested in Isaacson and Madsen (1976).

D. Long-Run Expected Average Annual Cost

Recall the assumptions made in Chapter I that demands occurring when the system is out of stock are backordered, units are demanded one at a time, and procurement lead time is constant τ. We don't need to place any additional condition on τ such as $\tau \leq T_k$ for $k = 0, 1, 2, \ldots$. The reason is that even if an order placed at the $(k-1)^{st}$ review time is not arrived until the next review time T_k, the decision on whether an order has to be placed at the time T_k will depend upon $\{IP_{T_{k-1}}\}$ and $\{D_{(T_{k-1},T_k]}\}$, where $D_{(T_{k-1},T_k]} \equiv N_{T_k} - N_{T_{k-1}}$ for $k = 1, 2, \ldots,$ with $\{N_t\}_{t=0}^{\infty}$ denoting a nonstationary Poisson process representing customer demands by time t. We want to make one additional assumption for these periodic-review systems that demands in different periods are independent random variables.

With the background about the long-run limit distributions of $\{IP_{T_k}\,;\; T_k \geq 0$ for $k = 0, 1, 2, \ldots\}$ discussed in the previous section, we are about to find the distribution of $\{NIS_{T_k+\xi}\}$ for $\xi \geq 0$ which can be immediately used to determine the expected on-hand inventory $E[OH_{T_k+\xi}]$ and the expected backorders $E[BO_{T_k+\xi}]$, at time $T_k + \xi$, in light of the following relations; by definition,

$$\begin{aligned} NIS_{T_k+\xi} &= IP_{T_k} - D_{(T_k,T_k+\xi]} \\ &= OH_{T_k+\xi} - BO_{T_k+\xi} \qquad (3.4.1) \end{aligned}$$

and hence,

$$NIS_{T_k+\xi} = CH_{T_k+\xi} , \quad \text{if} \quad NIS_{T_k+\xi} \geq 0$$
$$= BO_{T_k+\xi} , \quad \text{otherwise} . \qquad (3.4.2)$$

Then, using the cost factors discussed in Chapter II, we shall formulate cost functions under the periodic-review models, $<nQ, r, T>$ and $<R, r, T>$, the minimization of which is the criterion to determine the corresponding optimum operating policies.

1. The formulation of the $<nQ, r, T>$ model for the backorders case with nonstationary Poisson demands and constant lead times

Treating the inventory levels as discrete variable as well as the demand variable, we shall first prove that $\{IP_{T_k} ; T_k \geq 0$ for $k = 0, 1, 2, \ldots\}$ and $\{D_{(T_k, T_k+\xi]} ; T_k \geq 0$ for $k = 0, 1, 2, \ldots$ and $\xi \geq 0\}$ are mutually independent of each other.

Recall that under the $<nQ, r, T>$ operating doctrine, for $k = 0, 1, 2, \ldots$ an order is placed at a review time T_k if and only if the inventory position IP_{T_k} of the system is less than or equal to r. If $IP_{T_k} \leq r$, then a quantity nQ is ordered, where n is chosen such that $r < IP_{T_k} + nQ \leq r + Q$ for $n = 1, 2, 3, \ldots$. Thus, immediately after a review, the inventory position of the system will be in one of the Q states $r+1, r+2, \ldots, r+Q$. Therefore, the problem under this model is to determine the optimal values of Q, r and T which minimize the objective cost function to be derived later.

Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of nonnegative integer multipliers of Q for possible ordering at each review time T_i and $\{N_{ij}\}_{i=1}^{\infty}$ $(j = 1, 2, \ldots, Q)$ be a subsequence for which there exists $\{n_{1j}, n_{2j}, \ldots, n_{kj}\}$ associated with the ordering decisions which locate IP_{T_k} at a level $r+j$. Defining $M_{kj} = \sum_{i=1}^{k} n_{ij}$, $M_{kj} \cdot Q$ represents total amount of order placed by time T_k. If an inventory system starts with $IP_0 \equiv r+i$ $(i = 1, 2, \ldots, Q$, and $T_0 \equiv 0)$, then total amount stored in by time T_k on the books will be equal to $(r+i) + M_{kj} \cdot Q$. Since {total amount stored in on the books until immediately after the k^{th} review time T_k} minus {cumulative demand by T_k} equals to {inventory position immediately after T_k}, following equality follows;

$$\{(r+i) + M_{kj} \cdot Q\} - \{D_{(0,T_k]}\} = r+j, \quad (j = 1, 2, \ldots, Q).$$

$$\therefore \quad D_{(0,T_k]} = M_{kj} \cdot Q + (i-j), \quad \text{for } M_{kj} = 0, 1, 2, \ldots \tag{3.4.3}$$

In other words, $M_{kj} \cdot Q + (i-j)$ is the cumulative demand by time T_k which gets the inventory system to end up with $IP_{T_k} = r+j$.

Before proceeding to the next step, the following definition is needed; for $m = 0, 1, 2, \ldots$,

$$P\{D_{(0,T_k]} = i-j,\ D_{(T_k,T_k+\xi]} = m\}^{+} = P\{D_{(0,T_k]} = i-j\}^{+} P\{D_{(T_k,T_k+\xi]} = m\}$$

$$= P\{D_{(0,T_k]} = i-j\}\, P\{D_{(T_k,T_k+\xi]} = m\}, \quad \text{if } i \geq j \tag{3.4.4}$$

$= 0 ,$ otherwise .

Theorem III.D.1: For the periodic-review $<nQ, r, T>$ inventory system with nonstationary Poisson demand, constant lead time $\tau \geq 0$ and possible backorders, and also with $IP_0 \equiv r + i$ $(i = 1, 2, \ldots, Q)$,

$$P\{IP_{T_k} = r + j,\ D_{(T_k, T_k+\xi]}\} = P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = m\} ,$$

$$(m,k = 0, 1, 2, \ldots, \text{ and } j = 1, 2, \ldots, Q).$$

Proof:

By use of the same idea applied in Theorem II.C.4,

$$P\{IP_{T_k} = r + j,\ D_{(T_k, T_k+\xi]} = m\}$$

$$= \left[\begin{array}{l} P\{D_{(0,T_k]} = i - j,\ D_{(T_k, T_k+\xi]} = m\}^{+} \\ \\ + \sum_{M_{kj}=1}^{\infty} P\{D_{(T_k, T_k+\xi]} = m \mid D_{(0,T_k]} = M_{kj} \cdot Q + (i-j)\} \\ \\ \cdot\ P\{D_{(0,T_k]} = M_{kj} \cdot Q + (i - j)\} \end{array} \right]$$

$$= \left[\begin{array}{l} P\{D_{(0,T_k]} = i - j\}^{+}\ P\{D_{(T_k, T_k+\xi]} = m\} \\ \\ + \sum_{M_{kj}=1}^{\infty} P\{D_{(T_k, T_k+\xi]} = m\}\ P\{D_{(0,T_k]} = M_{kj} \cdot Q + (i - j)\} \end{array} \right]$$

$$= \left[P\{D_{(0,T_k]} = i - j\}^+ + \sum_{M_{kj}=1}^{\infty} P\{D_{(0,T_k]} = M_{kj} - Q + (i-j)\} \right]$$

$$\cdot\ P\{D_{(T_k, T_k+\xi]} = m\}$$

$$= P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = m\} \ .$$

Therefore, it is proved that $\{IP_{T_k}\}$ and $\{D_{(T_k, T_k+\xi]}\}$ are mutually independent of each other.

In order to compute the probability distribution of $\{NIS_{T_k+\xi};\ T_k \geq 0 \text{ and } \xi \geq 0\}_{k=0}^{\infty}$, we need define the following relation; for $j = 1, 2, \ldots, Q$,

$$P\{IP_{T_k},\ r + j,\ D_{(T_k, T_k+\xi]} = j - s\}^+ = P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = j - s\} \ , \quad \text{if} \quad j \geq s \qquad (3.4.5)$$

$$= 0 \ , \qquad \text{otherwise.}$$

Referring to Eq. (3.4.1),

$$P\{NIS_{T_k+\xi} = r + s\} = \sum_{j=1}^{Q} P\{IP_{T_k} = r + j,\ D_{(T_k, T_k+\xi]} = j - s\}^+$$

for $s = Q, Q-1, Q-2, \ldots$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = j - s\}^+$$

from Theorem III.D.1 and Eq. (3.4.5). (3.4.6)

If we assume that $\{IP_{T_k}\,;\ T_k \geq 0\}_{k=0}^{\infty}$ forms a weakly ergodic Markov chain, then its long-run limit distribution is uniform as shown in Eq. (3.3.5). If a demand process appearing in a cyclic fashion such as $P_{nd+\ell} = P_\ell$ ($\ell = 1, 2, \ldots, d;\ n = 0, 1, 2, \ldots$) is taken into account, the same uniform long-run distribution is also achieved in light of Theorem III.B.8, or Corollary III.B.1 and Theorem III.B.7 since for all k the k^{th} transition matrix shown in Table III.1 has at least one uniformly positive column (i.e., $\sigma^k \geq \sigma > 0$). Therefore, for the non-stationary Markov chain $\{IP_{T_k}\,;\ T_k \geq 0\}_{k=0}^{\infty}$ which is weakly ergodic or appears in a cyclic fashion, the same objective cost expression will be obtained in Eq. (3.4.20).

From Eq. (3.4.2) and Eq. (3.4.6),

$$P\{OH_{T_k+\xi} = x\} = P\{NIS_{T_k+\xi} = x\}\ , \quad \text{for} \quad x = 0, 1, 2, \ldots$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r + j,\ D_{(T_k, T_k+\xi]} = r + j - x\}^{+}$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = r + j - x\}^{+}\ . \qquad (3.4.7)$$

$$\therefore\ E[OH_{T_k+\xi}] = \sum_{x=0}^{\infty} x \cdot P\{OH_{T_k+\xi} = x\}$$

$$= \sum_{x=0}^{\infty} x \cdot \sum_{j=1}^{Q} P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = r + j - x\}^{+}$$

$$= \sum_{j=1}^{Q} \sum_{x=0}^{\infty} x \cdot P\{IP_{T_k} = r+j\} \, P\{D_{(T_k, T_k+\xi]} = r+j-x\}^{+}$$

$$= \sum_{j=1}^{Q} \sum_{x=0}^{r+j} x \cdot P\{IP_{T_k} = r+j\} \, P\{D_{(T_k, T_k+\xi]} = r+j-x\}$$

$$= \sum_{j=1}^{Q} \sum_{y=0}^{r+j} (r \cdot j - y) \, P\{IP_{T_k} = r+j\} \, P\{D_{(T_k, T_k+\xi]} = y\} \, ,$$

where $y = r + j - x$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r+j\} \Big[(r+j) \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\}$$

$$- \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\} \Big] \, . \qquad (3.4.8)$$

Thus, the long-run expected average number of unit years of on-hand inventory (storage) is

$$\lim_{k \to \infty} E[OH_{T_k+\xi}] = \lim_{k \to \infty} \sum_{j=1}^{Q} P\{IP_{T_k} = r+j\} \; [(r+j) \sum_{y=0}^{r+j} \cdot$$

$$P\{D_{(T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}]$$

$$= \sum_{j=1}^{Q} [\lim_{k \to \infty} P\{IP_{T_k} = r+j\}] \lim_{k \to \infty} [(r+j) \sum_{y=0}^{r+j} \cdot$$

$$P\{D_{T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}]$$

$$= \frac{1}{Q} \sum_{j=1}^{Q} \lim_{k \to \infty} \Big[(r+j) \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\} \Big] , \qquad (3.4.9)$$

from Eq. (3.3.5) .

Note: If a review takes place at time T_k , then the next review will take place at time T_{k+1} or $T_k + \Delta T_k$, where $\Delta T_k \equiv T_{k+1} - T_k$. Everything on order immediately after the review at time T_k will arrive in the system by $T_k + \tau$, but nothing not on order can arrive before time $T_{k+1} + \tau$ or $T_k + \Delta T_k + \tau$. Therefore, we shall consider the range of ξ between τ and $\Delta T_k + \tau$.

The long-run expected average number of unit years of on-hand inventory incurred per year, denoted by $E[OH]_{nQ}$, can be computed as follows;

$$E\,[OH]_{nQ} = \lim_{K \to \infty} \frac{\sum_{k=0}^{K} \frac{1}{\Delta T_k} \int_{\tau}^{\tau+\Delta T_k} E[OH_{T_k+\xi}]\, d\xi}{K} . \qquad (3.4.10)$$

We can also use the probability distribution of $\{NIS_{T_k+\xi}\}$ in Eq. (3.4.6) for computing $E[BO_{T_k+\xi}]$. From Eq. (3.4.2) and Eq. (3.4.6),

$$P\{BO_{T_k+\xi} = x\} = P\{NIS_{T_k+\xi} = -x\} \quad \text{for} \quad x = 1, 2, \ldots \quad \text{and} \quad \tau \le \xi \le \tau + \Delta T_k$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r+j,\ D_{(T_k, T_k+\xi]} = r+j+x\}^{+}$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r+j\}\ P\{D_{(T_k, T_k+\xi]} = r+j+x\} \qquad (3.4.11)$$

$$\therefore\ E[BO_{T_k+\xi}] = \sum_{x=1}^{\infty} x \cdot P\{BO_{T_k+\xi} = x\}$$

$$= \sum_{x=1}^{\infty} x \cdot [\sum_{x=1}^{Q} P\{IP_{T_k} = r+j\}\ P\{D_{(T_k, T_k+\xi]} = r+j+x\}]$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r+j\}\ [\sum_{x=1}^{\infty} x \cdot P\{D_{(T_k, T_k+\xi]} = r+j+x\}]$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r+j\} \sum_{j=r+j+1}^{\infty} (y-r-j)\ P\{D_{(T_k, T_k+\xi]} = y\}\ ,$$

where $y = r + j + x$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r+j\}\ [\sum_{y=r+j+1}^{\infty} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}$$

$$- (r+j) \sum_{j=r+j+1}^{\infty} P\{D_{(T_k, T_k+\xi]} = y\}]$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r+j\} \Big[E[D_{(T_k, T_k+\xi]}] - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}$$

$$- (r+j) \Big(1 - \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\}\Big)\Big]$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r + j\} \Big[E[D_{(T_k, T_k+\xi]}] - (r + j)$$

$$+ (r + j) \sum_{j=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\} \Big]$$

$$= \sum_{j=1}^{Q} P\{IP_{T_k} = r + j\} \Big[\int_{T_k}^{T_k+\xi} \lambda(u)du - (r + j)$$

$$+ (r + j) \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\} \Big],$$

since

$$E\,[D_{(T_k, T_k+\xi]}] = \int_{T_k}^{T_k+\xi} \lambda(u)du \qquad \text{from Eq. (3.3.2).} \qquad (3.4.12)$$

Under the weak ergodicity assumption on $\{IP_{T_k};\ T_k \geq 0\}_{k=0}^{\infty}$,

$$\lim_{k \to \infty} E\,[BO_{T_k+\xi}] = \sum_{j=1}^{Q} [\lim_{k \to \infty} P\{IP_{T_k} = r + j\}] \lim_{k \to \infty} \Big[\int_{T_k}^{T_k+\xi} \lambda(u)du$$

$$- (r+j) + (r+j) \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\} \Big]$$

$$- \sum_{j=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}$$

$$= \frac{1}{Q} \sum_{j=1}^{Q} \lim_{k \to \infty} \int_{T_k}^{T_k+\xi} \lambda(u)du - (r+j) + (r+j) \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\}$$

$$- \sum_{j=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\} \quad . \qquad (3.4.13)$$

Therefore, the long-run expected average number of unit years of backorders (or shortage) incurred per year, denoted by $E\,[BO]_{nQ}$, is given as follows;

$$E\,[BO]_{nQ} = \lim_{K \to \infty} \frac{\sum_{k=0}^{K} \frac{1}{\Delta T_k} \int_{\tau}^{\tau+\Delta T_k} E\,[BO_{T_k+\xi}]\, d\xi}{K} \quad . \qquad (3.4.14)$$

The random variable, say $\Delta BO_{T_k+\tau}$, representing the number of backorders incurred between $T_k+\tau$ and $T_{k+1}+\tau$ can be thought of as the difference between two random variables $BO_{T_k+\tau}$ and $BO_{T_{k+1}\tau}$ denoting the number of backorders on the books at time $T_k+\tau$ and $T_{k+1}+\tau$, respectively; that is, $\Delta BO_{T_k+\tau} = BO_{T_{k+1}+\tau} - BO_{T_k+\tau}$. Thus,

$$E\,[\Delta BO_{T_k+\tau}] = E\,[BO_{T_{k+1}+\tau} - BO_{T_k+\tau}]$$

$$= E\,[BO_{T_{k+1}+\tau}] - E\,[BO_{T_k+\tau}]$$

$$= \left(\sum_{j=1}^{Q} P\{IP_{T_{k+1}} = r+j\} \left[\int_{\tau}^{T_{k+1}+\tau} \lambda(u)du - (r+j) + (r+j) \right.\right.$$

$$\left.\left. \cdot \sum_{j=0}^{r+j} P\{D_{T_{k+1},T_{k+1}+\tau]} = y\} - \sum_{j=0}^{r+j} y \cdot P\{D_{(T_{k+1},T_{k+1}+\tau} = y\} \right] \right)$$

$$- \left(\sum_{j=1}^{Q} P\{IP_{T_k} = r+j\} \left[\int_{\tau}^{T_k+\tau} \lambda(u)du - (r+j) + (r+j) \right.\right.$$

$$\left.\left. \cdot \sum_{y=0}^{r+j} P\{D_{(T_k,T_k+\tau]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k,T_k+\tau]} = y\} \right] \right) . \qquad (3.4.15)$$

The long-run expected average number of backorders incurred between time $T_k + \tau$ and $T_{k+1} + \tau$ is

$$\lim_{k \to \infty} E[\Delta BO_{T_k+\tau}] = \lim_{k \to \infty} E[BO_{T_{k+1}+\tau}] - \lim_{k \to \infty} E[BO_{T_k+\tau}]$$

$$= \frac{1}{Q} \sum_{j=1}^{Q} \lim_{k \to \infty} \left[\left(\int_{T_{k+1}}^{T_{k+1}+\tau} \lambda(u)du - \int_{\tau}^{T_k+\tau} \lambda(u)du \right) + (r+j) \sum_{y=0}^{r+j} \right.$$

$$\cdot \left(P\{D_{(T_{k+1},T_{k+1}+\tau]} = y\} - P\{D_{(T_k,T_k+\tau]} = y\} \right)$$

$$\left. + \sum_{y=0}^{r+j} y \cdot \left(P\{D_{(T_k,T_k+\tau]} = y\} - P\{D_{(T_{k+1},T_{k+1}+\tau]} = y\} \right) \right] . \qquad (3.4.16)$$

Thus, the long-run expected average number of backorders incurred per year is

$$E\,[\Delta BO]_{nQ} = \lim_{K \to \infty} \frac{\sum_{k=0}^{K} E\,[\Delta BO_{T_k+\tau}]}{T_K+\tau}$$

$$= \lim_{K \to \infty} \frac{E\,[BO_{T_K+\tau}] - E\,[BO_\tau]}{T_K+\tau} \; . \qquad (3.4.17)$$

We now need to determine the expected review cost per period and then estimate the ensemble average to obtain the long-run expected average annual cost expression. Denote by W the cost of review. Since k reviews are made by time T_k , the long-run average annual cost of reviews is $\lim_{k \to \infty} \frac{k \cdot W}{T_k}$. With the cost A of placing an order, the average annual ordering cost can be determined if we know the probability P_{od} that an order will be placed at any given review time. Given that the inventory position of the system is $r+j$ immediately after a review, say T_k , then the probability that it will be less than or equal to r at the time of the next review T_{k+1} is the probability that j or more units are demanded during the review period $\Delta T_k \equiv T_{k+1} - T_k$; namely,

$$P\{IP_{T_{k+1}} \leq r \mid IP_{T_k} = r+j\} = P\{D_{(T_k, T_{k+1}]} \geq j \mid IP_{T_k} = r+j\}$$

for $j = 1, 2, \ldots, Q$ and $k = 0, 1, 2, \ldots$

$$= P\{D_{(T_k, T_{k+1}]} \geq j\} \quad \text{from Theorem III.D.1.} \qquad (3.4.18)$$

$$\therefore \quad P_{od} = \lim_{k \to \infty} P\{IP_{T_{k+1}} \leq r\} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

$$= \lim_{k \to \infty} \sum_{j=1}^{Q} P\{IP_{T_k} = r + j\} \, P\{IP_{T_{k+1}} \leq r \mid IP_{T_k} = r + j\}$$

$$= \lim_{k \to \infty} \sum_{j=1}^{Q} P\{IP_{T_k} = r + j\} \, P\{D_{(T_k, T_{k+1}]} \geq j\}$$

from Eq. (3.4.18)

$$= \frac{1}{Q} \sum_{j=1}^{Q} \lim_{k \to \infty} P\{D_{(T_k, T_{k+1}]} \geq j\} \quad . \qquad (3.4.19)$$

Therefore, the long-run average annual cost of placing orders is

$$\lim_{k \to \infty} \frac{A \cdot (k \cdot P_{od})}{T_k} \quad .$$

So far, we have evaluated all the terms needed in the cost expression. Hence, with the inventory carrying charge I, the unit cost of an item C, the fixed cost per unit backordered B and the cost per unit year of the shortage (backorders) $\hat{B}$ discussed in Chapter II, we can formulate the long-run expected average annual cost expression as follows;

$$\mathcal{L}(nQ, r, T) = \left(\lim_{k \to \infty} \frac{k \cdot W}{T_k} \right) + \left(\lim_{k \to \infty} \frac{A \cdot k \cdot P_{od}}{T_k} \right) + IC \cdot E\,[OH]_{nQ}$$

$$+ B \cdot E\,[\Delta BO]_{nQ} + \hat{B} \cdot E\,[BO]_{nQ} \quad . \qquad (3.4.20)$$

2. The formulation of the $<R, r, T>$ model for the backorders case with nonstationary Poisson demands and constant lead times

Recall that under the $<R, r, T>$ model (or an "Rr" operating doctrine), an order is placed immediately after the review time to bring the inventory position up to R, if the inventory position $\{IP_{T_k}\}$ at a review time T_k $(k = 0, 1, 2, \ldots)$ is less than or equal to r. Therefore, our objective under this model is to determine the optimal values of R, r, and $\Delta T_k \equiv T_{k+1} - T_k$ $(k = 0, 1, 2, \ldots)$ which minimize the inventory system operating cost. Thereby, we shall derive a cost function.

Under the same assumptions as those made for the $<nQ, r, T>$ case, it is known that an "Rr" operating policy is the optimal one, but the $<R, T>$ and the $<nQ, r, T>$ policies are only approximations to the optimal "Rr" doctrine.

We shall prove that $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ and $\{D_{(T_k, T_k+\xi]} ; T_k \geq 0; \xi \geq 0\}_{k=0}^{\infty}$ are mutually independent of each other. Let $\{OD_{\ell j}\}_{\ell=1}^{\infty}$ be a subsequence of $\{OD_{\ell}\}_{\ell=1}^{\infty}$ representing the amount of possible orders placed at each review time, for which there exists $\{OD_{1j}, OD_{2j}, \ldots, OD_{kj}\}$ leading the inventory system to have $\{IP_{T_k} = r+j\}$ for $j = 1, 2, \ldots, Q$. Assume that the inventory system starts with $IP_0 \equiv r+i$ $(i = 1, 2, \ldots, Q)$. Then, under the above assumptions the next equality follows;

$$\{(r+i) + ST_k\} - \{D_{(0,T_k]}\} = r + j , \tag{3.4.21}$$

where $ST_k \equiv \sum_{\ell=1}^{k} OD_{\ell j}$ denoting total amount stored in on the books until immediately after the k^{th} review in connection with $\{IP_{T_k} = r+j\}$ and having

$$\begin{cases} OD_\ell \geq R - r, & \text{if } D_{(T_{\ell-1}, T_\ell]} \geq R-r \quad (\ell = 1, 2, \ldots) \\ \quad = 0 & \text{otherwise} \end{cases}$$

$$\therefore \qquad D_{(0,T_k]} = ST_k + (i-j) \quad . \tag{3.4.22}$$

<u>Theorem III.D.2</u>: For the periodic-review $\langle R, r, T \rangle$ inventory system with the same restrictions placed in Theorem III.D.1,

$$P\{IP_{T_k} = r+j,\ D_{(T_k, T_k+\xi]} = m\} = P\{IP_{T_k} = r+j\}\ P\{D_{(T_k,T_k+\xi]} = m\} \ ,$$

$$(m,k = 0,\ 1,\ 2,\ \ldots;\ \ j = 1,\ 2,\ \ldots,\ Q).$$

Proof:

By the same approach applied in Theorem III.D.1,

$$P\{IP_{T_k} = r+j,\ D_{(T_k,T_k+\xi]} = m\}$$

$$= \left[\begin{array}{l} P\{D_{(0,T_k]} = i-j,\ D_{(T_k,T_k+\xi]} = m\}^{+} + \\ \\ \sum\limits_{ST_k=R-r}^{\infty} P\{D_{T_k,T_k+\xi]} = m \mid D_{(0,T_k]} = ST_k + (i-j)\}\ P\{D_{(0,T_k]} = ST_k + (i-j)\} \end{array} \right]$$

$$= \left[\begin{array}{l} P\{D_{(0,T_k]} = i-j\}^{+} P\{D_{(T_k,T_k+\xi]} = m\} \ + \\ \\ \sum_{ST_k=R-r}^{\infty} P\{D_{(T_k,T_k+\xi]} = m\} P\{D_{(0,T_k]} = ST_k + (i-j)\} \end{array} \right]$$

$$= [P\{D_{(0,T_k]} = i-j\}^{+} + \sum_{ST_k=R-r}^{\infty} P\{D_{(0,T_k]} = ST_k + (i-j)\}] \ P\{D_{(T_k,T_k+\xi]} = m\}$$

$$= P\{IP_{T_k} = r+j\} \ P\{D_{(T_k,T_k+\xi]} = m\} \ .$$

$\therefore$ It is proved that $\{IP_{T_k}\}$ and $\{D_{(T_k,T_k+\xi]}\}$ are mutually independent of each other.

The long-run limit distribution of $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ under the <R, r, T> case was discussed later in the previous section. The necessary and sufficient conditions were given in Theorem III.B.4 and in Theorem III.B.5. Theorem III.B.8 requires one necessary condition for a nonstationary Markov chain, appearing in a cyclic pattern such as $P_{nd+\ell} = P_{\ell}$ $(\ell = 1, 2, \ldots, d; \ n = 0, 1, 2, \ldots)$, to be strongly ergodic. It was shown in the previous section that the long-run limit distribution of a nonstationary Markov chain $\{IP_{T_k}\}_{k=0}^{\infty}$ associated with nonhomogeneous Poisson demands appearing in a cyclic fashion can be easily estimated by using Theorem III.B.8, since for all k the k^{th} transition matrix in Table III.2 has at least one uniformly positive column (see Eq. (3.3.7)) and hence the stationary matrices R_{ℓ} $(\ell = 1, 2, \ldots, d)$ are strongly ergodic.

Under the assumption that the finite long-run limit distribution of $\{IP_{T_k}\}$ is achievable, we shall derive the long-run expected average annual cost expression. Using Theorem III.D.2 and the relations of Eq. (3.4.1) and Eq. (3.4.5),

$$P\{NIS_{T_k+\xi} = r + s\} = \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j,\ D_{(T_k, T_k+\xi]} = j - s\}^+ ,$$

$$\text{for } S = R\text{-}r,\ R\text{-}r\text{-}1,\ \ldots,\ 0,\ -1,\ -2$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = j - s\}^+ ,$$

$$\text{for } \tau \leq \xi \leq \tau + \Delta T_k . \qquad (3.4.23)$$

From Eq. (3.4.2) and Eq. (3.4.23),

$$P\{OH_{T_k+\xi} = x\} = P\{NIS_{T_k+\xi} = x\} \qquad \text{for } x = 0, 1, 2, \ldots$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j,\ D_{(T_k, T_k+\xi]} = r + j - x\}^+$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j\}\ P\{D_{(T_k, T_k+\xi]} = r + j - x\}^+ . \qquad (3.4.24)$$

$$\therefore\ E\,[OH_{T_k+\xi}] = \sum_{x=0}^{\infty} x \cdot P\{OH_{T_k+\xi} = x\}$$

$$= \sum_{x=0}^{\infty} x \cdot P\{NIS_{T_k+\xi} = x\}$$

$$= \sum_{x=0}^{\infty} x \cdot \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j\} \; P\{D_{(T_k, T_k+\xi]} = r + j - x\}^{+}$$

$$= \sum_{j=1}^{R-r} \sum_{x=0}^{\infty} x \cdot P\{IP_{T_k} = r + j\} \; P\{D_{(T_k, T_k+\xi]} = r + j - x\}^{+}$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j\} \sum_{x=0}^{r+j} x \cdot P\{D_{(T_k, T_k+\xi]} = r + j - x\}$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j\} \sum_{y=0}^{r+j} (r + j - y) \; P\{D_{(T_k, T_k+\xi]} = y\} ,$$

where $y = r + j - x$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j\} \; [(r+j) \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\}$$

$$- \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}] \; . \quad (3.4.25)$$

Therefore, the long-run expected number of unit years of on-hand inventory (storage) is

$$\lim_{k \to \infty} E\,[OH_{T_k+\xi}] = \lim_{k \to \infty} \sum_{j=1}^{R-r} P\{IP_{T_k} = r + j\} \cdot [(r+j) \sum_{y=0}^{r+j}$$

$$\cdot \; P\{D_{(T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}]$$

$$= \sum_{j=1}^{R-r} [\lim_{k \to \infty} P\{IP_{T_{T_k}} = r+j\}] \lim_{k \to \infty} [(r+j) \sum_{y=0}^{r+j}$$

$$\cdot P\{D_{(T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\}] \cdot \qquad (3.4.26)$$

Thus, the long-run expected average number of unit years of on-hand inventory incurred per year, denoted by $E[OH]_R$, follows;

$$E[OH]_R = \lim_{K \to \infty} \frac{\sum_{k=0}^{K} \frac{1}{\Delta T_k} \int_{\tau}^{\tau+\Delta T_k} E[OH_{T_k+\xi}] \, d\xi}{K} \cdot \qquad (3.4.27)$$

Likewise, using Eq. (3.4.6) and the relation of $BO_{T_k+\xi}$ with $NIS_{T_k+\xi}$ in Eq. (3.4.2),

$$P\{BO_{T_k+\xi} = x\} = P\{NIS_{T_k+\xi} = -x\} \quad \text{for} \quad x = 1, 2, \ldots \quad \text{and} \quad \tau \le \xi \le \tau+\Delta T_k$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j, D_{(T_k, T_k+\xi]} = r+j+x\}^{+}$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\} P\{D_{(T_k, T_k+\xi]} = r+j+x\} \qquad (3.4.28)$$

$$\therefore \quad E[BO_{T_k+\xi}] = \sum_{x=1}^{\infty} x \cdot P\{BO_{T_k+\xi} = x\}$$

$$= \sum_{x=1}^{\infty} x \cdot \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\} \, P\{D_{(T_k, T_k+\xi]} = r+j+x\}$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\} \sum_{x=1}^{\infty} x \cdot P\{D_{(T_k, T_k+\xi]} = r+j+x\}$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\} \sum_{y=r+j+1}^{\infty} (y-r-j) \, P\{D_{(T_k, T_k+\xi]} = y\} \text{ , where } y = r+j+x$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\} \Bigg[E\,[D_{(T_k, T_k+\xi]}] - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\} - (r+j) \left(1 - \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\} \right) \Bigg]$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\} \Bigg[\int_{T_k}^{T_k+\xi} \lambda(u)du - (r+j) + (r+j) \sum_{y=0}^{r+j} P\{D_{(T_k, T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k, T_k+\xi]} = y\} \Bigg],$$

since

$$E\,[D_{(T_k, T_k+\xi]}] = \int_{T_k}^{T_k+\xi} \lambda(y)du, \text{ from Eq. (3.3.2).} \qquad (3.4.29)$$

Assuming that $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ is strongly ergodic,

$$\lim_{k \to \infty} E\left[BO_{T_k+\xi}\right] = \sum_{j=1}^{R-r}\left[\lim_{k\to\infty} P\{IP_{T_{T_k}} = r+j\}\right]$$

$$\cdot \lim_{k\to\infty}\left[\int_{T_k}^{T_k+\xi} \lambda(u)du - (r+j) + (r+j)\sum_{y=0}^{r+j} P\{D_{(T_k,T_k+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k,T_k+\xi]} = y\}\right] . \quad (3.4.30)$$

Thus, the long-run expected average number of unit years of backorders incurred per year, denoted by $E[BO]_R$, is

$$E\left[BO\right]_R = \lim_{K\to\infty} \frac{\sum_{k=0}^{K} \frac{1}{\Delta T_k} \int_{\tau}^{\tau+\Delta T_k} E\left[BO_{T_k+\xi}\right] d\xi}{K} . \quad (3.4.31)$$

Also,

$$E\left[\Delta BO_{T_k+\tau}\right] = E\left[BO_{T_{k+1}+\tau} - BO_{T_k+\tau}\right]$$

$$= E\left[BO_{T_{k+1}+\tau}\right] - E\left[BO_{T_k+\tau}\right] \quad (3.4.32)$$

where

$$E\left[BO_{T_k+\tau}\right] = \sum_{j=1}^{R-r} P\{IP_{T_{T_k}} = r+j\}\left[\int_{T_k}^{T_k+\tau} \lambda(u)du - (r+j) + \right.$$

$$(r+j)\sum_{y=0}^{r+j} P\{D_{(T_k,T_k+\tau]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_k,T_k+\tau]} = y\}\Bigg] .$$

The long-run expected number of backorders incurred between time $T_k+\tau$ and $T_{k+1}+\tau$ is

$$\lim_{k\to\infty} E\,[\Delta BO_{T_k+\tau}] = \lim_{k\to\infty} E\,[BO_{T_{k+1}+\tau}] - \lim_{k\to\infty} E\,[BO_{T_k+\tau}] . \tag{3.4.33}$$

Thus, the long-run expected average number of backorders incurred per year, denoted by $E\,[\Delta BO]_R$, is

$$E\,[\Delta BO]_R = \lim_{k\to\infty} \frac{\sum_{k=1}^{K} E\,[\Delta BO_{T_k+\tau}]}{T_K+\tau}$$

$$= \lim_{k\to\infty} \frac{E\,[BO_{T_k+\tau}] - E\,[BO_{\tau}]}{T_k+\tau} . \tag{3.4.34}$$

The probability P_{od} of an order being placed at a given review time can also be computed by the same approach in Eq. (3.4.19);

$$P_{od} = \lim_{k\to\infty} P\{IP_{T_{k+1}} \le r\} \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

$$= \lim_{k\to\infty} \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\}\, P\{IP_{T_{k+1}} \le r \mid IP_{T_k} = r+j\}$$

$$= \lim_{k \to \infty} \sum_{j=1}^{R-r} P\{IP_{T_k} = r+j\} \, P\{D_{(T_k,\, T_{k+1}]} \geq j\} \ ,$$

since from Theorem III.D.2

$$P\{IP_{T_{k+1}} \leq r \mid IP_{T_k} = r+j\}$$

$$= P\{D_{(T_k, T_{k+1}]} \geq j \mid IP_{T_k} = r+j\}$$

$$= P\{D_{(T_k, T_{k+1}]} \geq j\}$$

$$= \sum_{j=1}^{R-r} [\lim_{k \to \infty} P\{IP_{T_k} = r+j\}] \, [\lim_{k \to \infty} P\{D_{(T_k, T_{k+1}]} \geq j\}] \ . \tag{3.4.35}$$

Hence, with the review cost W the long-run expected average annual cost function can be given as follows;

$$\mathcal{L}(R, r, T) = (\lim_{k \to \infty} \frac{k \cdot W}{T_k}) + (\lim_{k \to \infty} \frac{A \cdot k \cdot P_{od}}{T_k})$$

$$+ B \cdot E[\Delta BO]_R + \hat{B} \cdot E[BO]_R \ . \tag{3.4.36}$$

The rest of this section will cover the derivation of the long-run expected average annual cost expression in the case of $\{IP_{T_k};\ T_k \geq 0\}_{k=0}^{\infty}$

associated with nonhomogeneous Poisson demands appearing in a cyclic pattern.

Let $\{P_k\}_{k=0}^{\infty}$ be the transition probability matrices of a cyclic nonstationary Markov chain $\{IP_{T_k}\}_{k=0}^{\infty}$ and repeat themselves such as $P_{nd+\ell} = P_\ell$ $(\ell = 1, 2, \ldots, d;\ n = 0, 1, 2, \ldots)$, where $k = nd+\ell$. Denote by $\pi_\ell = (g_{\ell,r+1}, g_{\ell,r+2}, \ldots, g_{\ell,R})$ the finite row vector of a constant matrix G_ℓ for $\ell = 1, 2, \ldots, d$. Then, according to Theorem III.B.8, π_ℓ can be defined as the left eigenvector of stationary finite matrix R_ℓ corresponding to eigenvalue 1 and hence the long-run limit distribution of $\{IP_{T_{nd+\ell}}\}$ as $n \rightarrow \infty$. Therefore, in the approach to the $\langle R, r, T\rangle$ case which takes into account the cyclic behavior of stochastic demands, we can also derive the relevant expected average annual cost expression similar to the above work.

Using Theorem III.D.2 and the relations of Eq. (3.4.1) and Eq. (3.4.5),

$$P\{NIS_{T_{nd+\ell}+\xi} = r+s\} = \sum_{j=1}^{R-r} P\{IP_{T_{nd+\ell}} = r+j\}\ P\{D_{(T_{nd+\ell}, T_{nd+\ell}+\xi]} = j-s\}^{+} \tag{3.4.37}$$

$$\text{for } \tau \leq \xi \leq \tau + \Delta T_{nd+\ell}, \text{ where } \Delta T_{nd+\ell} \equiv T_{nd+\ell+1} - T_{nd+\ell}.$$

From Eq. (3.4.2) and Eq. (3.4.37),

$$P\{OH_{T_{nd+\ell}+\xi} = x\} = P\{NIS_{T_{nd+\ell}+\xi} = x\} \qquad \text{for} \quad x = 0, 1, 2, \ldots$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_{nd+\ell}} = r+j\}\ P\{D_{(T_{nd+\ell},\ T_{nd+\ell}+\xi]} = r+j-x\}^{+}. \qquad (3.4.38)$$

From Eq. (3.4.25),

$$E\,[OH_{T_{nd+\ell}+\xi}] = \sum_{x=0}^{\infty} x \cdot P\{OH_{T_{nd+\ell}+\xi} = x\}$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_{nd+\ell}} = r+j\} \left[(r+j) \sum_{y=0}^{r+j} P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = y\} \right] . \qquad (3.4.39)$$

Therefore, the long-run expected number of unit years of on-hand inventory (storage) is

$$\lim_{n \to \infty} E\,[OH_{T_{nd+\ell}+\xi}] = \sum_{j=1}^{R-r} [\lim_{n \to \infty} P\{IP_{T_{nd+\ell}} = r+j\}]$$

$$\cdot \lim_{n \to \infty} [(r+j) \sum_{y=0}^{r+j} P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = y\}$$

$$- \sum_{y=0}^{r+j} y \cdot P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = y\}]$$

$$= \sum_{j=1}^{R-r} g_{\ell,j} \lim_{n \to \infty} \left[(r+j) \sum_{y=0}^{r+j} P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = y\} \right. -$$

$$\left. \sum_{y=0}^{r+j} y \cdot P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = y\} \right] , \qquad (3.4.40)$$

where

$$P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = y\} = \frac{e^{-\int_{T_{nd+\ell}}^{T_{nd+\ell}+\xi} \lambda(u)dy} \left[\int_{T_{nd+\ell}}^{T_{nd+\ell}+\xi} \lambda(u)du \right]^{y}}{y!} .$$

Thus, the long-run expected average number of units years of on-hand inventory incurred per year, denoted by $E\,[OH]_{RC}$, is

$$E\,[OH]_{RC} = \lim_{N \to \infty} \frac{\sum_{n=0}^{N} \sum_{\ell=1}^{d} \frac{1}{\Delta T_{nd+\ell}} \int_{\tau}^{T+\Delta T_{nd+\ell}} E\,[OH_{T_{nd+\ell}+\xi}]d\xi}{N} , \qquad (3.4.41)$$

where

$$\Delta T_{nd+\ell} \equiv T_{nd+\ell+1} - T_{nd+\ell} .$$

Similarly, using Eq. (3.4.2), Eq. (3.4.6), and Eq. (3.4.28),

$$P\{BO_{T_{nd+\ell}+\xi} = x\} = P\{NIS_{T_{nd+\ell}+\xi} = -x\} \quad \text{for} \quad x = 0, 2, \ldots \qquad (3.4.42)$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_{nd+\ell}} = r+j\} \; P\{D_{(T_{nd+\ell},T_{nd+\ell}+\xi]} = r+j+x\} .$$

$$\therefore E\,[BO_{T_{nd+\ell}+\xi}] = \sum_{x=1}^{\infty} x \cdot P\{BO_{T_{nd+\ell}+\xi} = x\}$$

$$= \sum_{j=1}^{R-r} P\{IP_{T_{nd+\ell}} = r+j\} \left[\int_{T_{nd+\ell}}^{T_{nd+\ell}+\xi} \lambda(u)du - (r+j) \right.$$

$$+ (r+j) \sum_{y=0}^{r+j} P\{D_{(T_{nd+\ell}, T_{nd+\ell}+\xi]} = y\}$$

$$\left. - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_{nd+\ell}, T_{nd+\ell}+\xi]} = y\} \right] , \qquad (3.4.43)$$

from Eq. (3.4.29).

Therefore, the long-run expected number of units years of backorders (shortage) is

$$\lim_{n \to \infty} E\,[BO_{T_{nd+\ell}+\xi}]$$

$$= \sum_{j=1}^{R-r} \left[\lim_{n \to \infty} P\{IP_{T_{nd+\ell}} = r+j\} \right] \lim_{n \to \infty} \left[\int_{T_{nd+\ell}}^{T_{nd+\ell}+\xi} \lambda(u)du - (r+j) \right.$$

$$\left. + (r+j) \sum_{y=0}^{r+j} P\{D_{(T_{nd+\ell}, T_{nd+\ell}+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_{nd+\ell}, T_{nd+\ell}+\xi]} = y\} \right]$$

$$= \sum_{j=1}^{R-r} g_{\ell,j} \lim_{n \to \infty} \left[\int_{T_{nd+\ell}}^{T_{nd+\ell}+\xi} \lambda(u)du - (r+j) + \right.$$

$$(r+j) \sum_{y=0}^{r+j} P\{D_{(T_{nd+\ell}, T_{nd+\ell}+\xi]} = y\} - \sum_{y=0}^{r+j} y \cdot P\{D_{(T_{nd+\ell}, T_{nd+\ell}+\xi]} = y\} \Bigg] . \tag{3.4.44}$$

Thus, the long-run expected average number of unit years of backorders incurred per year, denoted by $E\,[BO]_{RC}$, is

$$E\,[BO]_{RC} = \lim_{N \to \infty} \frac{\sum_{n=1}^{N} \sum_{\ell=1}^{d} \frac{1}{\Delta T_{nd+\ell}} \int_{\tau}^{\tau+\Delta T_{nd+\ell}} E\,[BO_{T_{nd+\ell}+\xi}]\,d\xi}{N} . \tag{3.4.45}$$

From Eq. (3.4.32) and Eq. (3.4.33), the long-run expected number of backorders incurred between time $T_{nd+\ell}+\tau$ and $T_{nd+\ell+1}+\tau$ is

$$\lim_{n \to \infty} E\,[\Delta BO_{T_{nd+\ell}+\tau}] = \lim_{n \to \infty} E\,[BO_{T_{nd+\ell+1}+\tau}] - \lim_{n \to \infty} E[BO_{T_{nd+\ell}+\tau}] . \tag{3.4.46}$$

Therefore, the long-run expected average number of backorders incurred per year, denoted by $E\,[\Delta BO]_{RC}$, is

$$E\,[\Delta BO]_{RC} = \lim_{N \to \infty} \frac{\sum_{n=0}^{N} \sum_{\ell=1}^{d} E\,[\Delta BO_{T_{nd+\ell}+\tau}]}{T_{Nd}+\tau}$$

$$= \lim_{N \to \infty} \frac{E\,[BO_{T_{Nd}+\tau}] - E\,[BO_{\tau}]}{T_{Nd}+\tau} . \tag{3.4.47}$$

From Eq. (3.4.35),

$$P_{od} = \lim_{n \to \infty} P\{IP_{T_{nd+\ell+1}} \leq r\} \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

$$= \sum_{j=1}^{R-r} [\lim_{n \to \infty} P\{IP_{T_{nd+\ell}} = r + j\} \lim_{n \to \infty} P\{D(T_{nd+\ell}, T_{nd+\ell+1}] \geq j\}$$

$$= \sum_{j=1}^{R-r} g_{\ell,j} \lim_{n \to \infty} P\{D(T_{nd+\ell}, T_{nd+\ell+1}] \geq j\} \quad . \tag{3.4.48}$$

Hence, with the review cost W the long-run expected average annual cost function in the cyclic $<R, r, T>$ case is formulated as follows;

$$\mathcal{L}(R, r, T)_C = (\lim_{N \to \infty} \frac{Nd \cdot W}{T_{Nd}}) + (\lim_{N \to \infty} \frac{A \cdot (Nd) \cdot P_{od}}{T_{Nd}}) + IC \cdot E\,[OH]_{RC}$$

$$+ B \cdot E\,[\Delta BO]_{RC} + \hat{B} \cdot E\,[BO]_{RC} \; . \tag{3.4.49}$$

IV. SUMMARY AND CONCLUSION

This study has aimed at the analysis of nonstandard inventory models, with general iid inter-demand times for transactions reporting, and nonstationary Markov demand for periodic review.

The subject has been developed in the context of the case in which demands occurring when the system is out of stock are backordered, units are demanded one at a time, and procurement lead time is constant. The inventory systems under study were assumed to consist of just one stocking point with a single source of resupply. The relevant cost parameters involved in the objective cost expressions were assumed to be in stationary variations with time.

Under the above assumptions, the cumulative demand by time t, $\{N_t; t \geq 0\}$, is a discrete-valued continuous-parameter stochastic process (a renewal process) with sample paths increasing in unit steps. $\{N_t\}$ was analyzed first to describe probabilistically the inventory position $\{IP_t; t \geq 0\}$, under the $<Q, r>$ model for transactions reporting, and under the $<nQ, r, T>$ and $<R, r, T>$ models for periodic review. It was shown that $\{IP_t; t \geq 0\}$ totally depends on $\{N_t; t \geq 0\}$ and an initial inventory position $\{IP_0\}$.

In the case of the $<Q, r>$ model, the relation between $\{N_t\}$ and the n^{th} renewal time S_n, where $N_t \equiv \sup\{n; S_n \leq t\}$, played a key role to prove that $\{IP_{t-\tau}\}$ and the cumulative demand between time $t-\tau$ and t, $\{D_{(t-\tau,t]}\}$, are mutually independent of each other (see Theorem II.C.4). Corollary II.B.1 developed during this study and Key

Renewal Theorem II.B.6 were applied to the computation of the asymptotic limit distributions of $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ in, respectively, Theorem II.D.1 and Theorem II.D.3. Then, the joint distribution of $\{IP_{t-\tau}\}$ and $\{D_{(t-\tau,t]}\}$ was formulated in Theorem II.C.4. Once the distribution of NIS_t in Eq. (2.5.3) was determined by use of the joint distribution, the distributions of the on-hand inventory $\{OH_t\}$ and the backorders $\{BO_t\}$ were easily computed in, respectively, Eq. (2.5.4) and Eq. (2.5.8). Thus, the evaluation of their long-run expected values $E[OH]_Q$ in Eq. (2.5.7) and $E[BO]_Q$ in Eq. (2.5.11), which were necessary for the long-run expected average annual cost expression, was straightforward. Finally, the cost expression for the $\langle Q, r\rangle$ model was derived in Eq. (2.5.15).

In addition to the assumptions mentioned above, one more assumption was added in the case of periodic review inventory systems, i.e., that demands in different review periods are independent random variables.

In the case of the $\langle nQ, r, T\rangle$ model, Theorem III.B.7 was applied to determine that the long-run distribution of $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ associated with the nonstationary Poisson demand process $\{D_{(T_k,T_{k+1}]} ; T_k \geq 0\}_{k=0}^{\infty}$ is uniform when $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ is weakly ergodic, where $(T_k, T_{k+1}]$ represents the $(k+1)^{st}$ review period with $T_0 \equiv 0$. However, if the cumulative demand process $\{N_t ; t \geq 0\}$ appears in a cyclic pattern for which the corresponding transition probability matrices P_k of $\{IP_{T_k}\}_{k=0}^{\infty}$ repeat themselves in a cyclic fashion (that is, $P_{nd+\ell} = P_\ell$ for $\ell = 1, 2, \ldots, d$ and $n = 0, 1, 2, \ldots$) , then the long-run limit distribution of $\{IP_{T_k}\}$ is uniform. This result was shown in Eq. (3.3.6).

Thus, we came to the conclusion that Theorem III.B.7 is robust for this <nQ, r, T> model, because whatever the demand distributions are they will be formed into the corresponding doubly stochastic matrices for $\{IP_{T_k} ; T_k \geq 0\}_{k=0}^{\infty}$ and thence the uniform limit distribution will be ended up with. The uniformity leads to the standard computation of expected cost. In Theorem III.D.1, it was proved that $\{IP_{T_k}\}$ and $\{D_{(T_k, T_k+\xi]} ; \xi \geq 0\}$ $(k = 0, 1, 2, \ldots)$ are mutually independent of each other. Given the result in Theorem III.D.1, the distribution of $\{NIS_{T_k+\xi}\}$ was derived in Eq. (3.4.6), and hence the long-run expected average values of $E\,[OH]_{nQ}$ and $E\,[BO]_{nQ}$ were evaluated in, respectively, Eq. (3.4.10) and Eq. (3.4.14). The probability, P_{od} , that an order will be placed at any given review time was also taken into account in formulating the long-run expected average annual cost expression in Eq. (3.4.19).

In the case of the <R, r, T> model, conditions from nonstationary Markov Chain Theory were given in Theorem III.B.4 and Theorem III.B.5 which, together with an easily verified condition for weak ergodicity in Theorem III.B.4 and Corollary III.B.1, are sufficient for the distributional convergence of $\{IP_{T_k}\}$, and hence of $\{NIS_{T_k+\xi}\}$ for $\xi \geq 0$ and $k = 0, 1, 2, \ldots$ Theorem III.D.2 proved that under the model, $\{IP_{T_k}\}$ and $\{D_{(T_k, T_k+\xi]} ; \xi \geq 0\}$ $(k = 0, 1, 2, \ldots)$ are also mutually independent of each other. Thence, the distribution of $\{NIS_{T_k+\xi}\}$ was determined in Eq. (3.4.23), and $E\,[OH]_R$ and $E\,[BO]_R$ were evaluated in, respectively, Eq. (3.4.27) and Eq. (3.4.31). The long-run expected average annual cost expression was finally derived in Eq. (3.4.34).

This study also included an approach to the $\langle R, r, T \rangle$ case which takes into account possible cyclic behavior of demand. The result of Eq. (3.3.7) indicates that the strong ergodicity condition of R_d in Theorem III.B.8 is satisfied, since R_4 turns out a stationary finite primitive matrix. Therefore, the long-run limit distribution of subsequences $\{IP_{T_{nd+\ell}}\}$ is the row vector of the corresponding limit matrix $\{G_\ell\}$ for $\ell = 1, 2, \ldots, d$ and $n = 0, 1, 2, \ldots$. The corresponding cost function was derived in Eq. (3.4.49).

In the case of the $\langle R, r, T \rangle$ model with stationary Poisson demand studied in Hadley and Whitin (1963), the simpler closed form of solutions for the long-run limit distribution of $\{IP_{kT};\ T \geq 0\}$ $(k = 0, 1, 2, \ldots)$ was derived in terms of recurrence coefficients in Eq. (3.3.11).

The application of nonstationary Markov Chain Theory to periodic-review inventory control is a more realistic and better approach, and also relatively easy to make because the finite transition matrices are involved in the determination of long-run limit distribution. Once having derived the cost functions $\mathcal{L}$, Mathematical Programming (including Dynamic Programming) technique will be required to determine the relevant optimal values of Q, r, R and T on a Digital computer which minimize $\mathcal{L}$.

The <R, T> model is a special case of the <nQ, r, T> model with Q = 1 and R = r + 1 . Therefore, once having obtained the equations for the <nQ, r, T> model, the derivation of the cost function for the <R, T> model will be straightforward under the same assumptions which apply in deriving the <nQ, r, T> model.

A. Further Research

The extension of this study to other inventory systems with random lead time or with random demand units will be an immediate challenge. For example, if we assume that lead times are independent and the range of the times is restricted less than T , where $T \equiv \min \Delta T_k$ (k = 0, 1, 2, ...), and orders are received in the same sequence in which they were placed, then the inclusion of stochastic lead times will be allowed in the periodic-review models developed in Chapter III by accounting for the distribution of lead times as follows; for example,

$$E\,[BO]_{nQ} = \lim_{K \to \infty} \frac{\sum_{k=0}^{K} \frac{1}{\Delta T_k} \int_0^{\Delta T_{k+1}} \int_0^{\Delta T_k} \int_{\tau_k}^{T_{k+1}+\Delta T_k} E\,[BO_{T_k+\xi}]d\xi\, f_{L_k}(\tau_k) \cdot f_{L_{k+1}}(\tau_{k+1}) d\tau_k d\tau_{k+1}}{K}$$

where

$$\begin{cases} f_{L_k}(\cdot) = \text{the probability density function of the random lead time } L_k \text{ of the order placed at time } T_k \text{ such that } L_k \le \Delta T_k , \\ f_{L_{k+1}}(\cdot) = \text{the probability density function of } L_{k+1} \text{ of the order placed at time } T_{k+1} \text{ such that } L_{k+1} \le \Delta T_{k+1} . \end{cases}$$

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